Scaling regimes of the $2d$ Navier–Stokes equation with self similar stirring

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Introduction

Velocity and Vorticity in $2d$

The Navier Stokes equation

$$(\partial_t + v \cdot \partial) v^\alpha - \nu \, \partial^2 v^\alpha = -\partial^\alpha P + f^\alpha \,, \qquad \alpha = 1, 2$$

$$\partial \cdot v = 0$$

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in *two dimensions* transports the vorticity field

$$\omega := \epsilon_{\alpha\,\beta} \partial^\alpha \, v^\beta \, , \qquad \omega = \text{vorticity}$$
$$\partial_t \omega + v \cdot \partial \omega - \nu \, \partial^2 \omega = \epsilon_{\alpha\,\beta} \partial^\alpha f^\beta$$

Velocity and Vorticity in $2d$

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Conservation of vorticity moments in the inviscid limit.

Kraichnan's picture of 2d turbulence

Kàrmàn-Howarth-Monin equation

Gaussian, time short-correlated translational invariant forcing

$$\prec f^{\alpha}(x,t)f_{\alpha}(y,s)\succ = \delta(t-s)\,F(x-y,m)$$

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$$\delta v(x) := v(x,t) - v(0,t)$$

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the Kàrmàn-Howarth-Monin (KHM) equation (equal times) is

$$\frac{1}{2} \prec (\partial \cdot \delta v)(x) \delta v(x)^2 \succ =$$

$$= \partial_t \prec v(x) \cdot v(0) \succ - F(x,m) - 2\,\nu\, \prec (\partial_{\alpha} v)(x) (\partial^{\alpha} v)(0) \succ$$

Hypotheses encoding Kraichnan's theory:

i velocity correlations are smooth at finite viscosity and exist in the inviscid limit even at coinciding points:

$$\lim_{x \to 0} \prec v(x) \cdot v(0) \succ = \prec v^2(0) \succ \qquad \nu > 0$$

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ii Galilean invariant functions and in particular structure functions reach a steady state:

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ii Galilean invariant functions and in particular structure functions reach a steady state:

$$\lim_{t\uparrow\infty} \prec \delta v^{\mu}(x)\delta v^2(x) \succ = S_3^{\mu}(x)$$

iii No energy dissipative anomalies occur:

$$\left\{ \lim_{\nu\downarrow 0}\lim_{x\downarrow 0} - \lim_{x\downarrow 0}\lim_{\nu\downarrow 0} \right\} \nu \prec \partial_{\alpha} v^{\beta}(x,t)\partial^{\alpha} v_{\beta}(0,t) \succ = 0$$

KHM equation and "mean field" scaling

$$\frac{1}{2}\partial_\mu S_3^\mu = \partial_t \prec v(x)\cdot v(0) \succ - F(x,m) - 2\,\nu \prec (\partial_\alpha v)(x)(\partial^\alpha v)(0) \succ$$

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Inverse Energy Cascade

$$\prec \delta v_{||}^3 \succ = \prec \delta v_{||}\delta v_\perp^2 \succ \stackrel{mx\gg 1}{=} \frac{3\, I_{\mathcal{E}}\, x}{2} \quad \stackrel{\text{mean field}}{\Rightarrow} \quad \delta v \sim x^{1/3}$$

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Direct Enstrophy Cascade

$$\prec \delta v_{||}^3 \succ = \prec \delta v_{||} \delta v_\perp^2 \succ \stackrel{\ell x \ll mx \ll 1}{=} \frac{I_\Omega\, x^3}{8} \quad \stackrel{\text{mean field}}{\Rightarrow} \quad \delta v \sim x$$

Energy spectrum

$$\mathcal{E}(k) = \int \frac{d^d p}{(2\pi)^d} \delta(|k| - |p|) \int d^d x \, e^{\imath p \cdot x} \prec v(x) \cdot v(0) \succ$$

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G. Boffetta,

J. Fluid Mech.

589, 253 (2007).

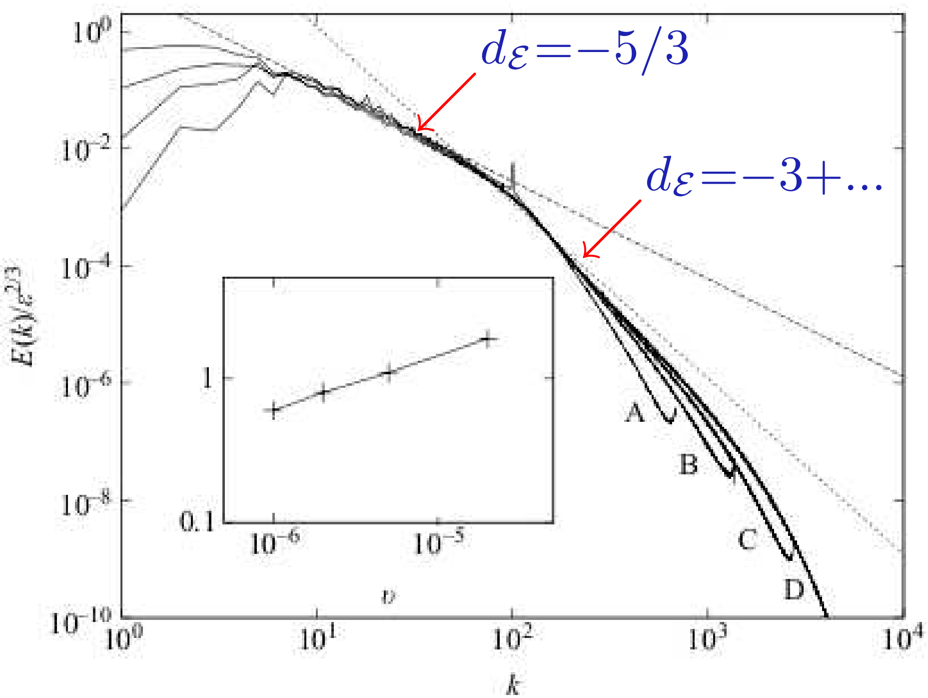

FIGURE 2. Energy spectra for the two simulations for the different resolutions (labels as in figure 1). Dashed and dotted lines represent the two predictions $Ck^{-5/3}$ with $C = 6$ and k^{-3} respectively. Inset: correction δ to the Kraichnan exponent -3 as a function of viscosity, computed by fitting the spectra with a power law $k^{-(3+\delta)}$ in the range $100 \leqslant k \leqslant 400$.

UV Renormalization Group analysis

Honkonen et al. (1998)

Power law forcing

$$(\partial_t + v \cdot \partial) v^\alpha - \nu\, \partial^2 v^\alpha = -\partial^\alpha P + f^\alpha - \frac{v^\alpha}{\tau} \qquad \alpha = 1, 2$$

Power law forcing

Ekman friction

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$$\prec f^\alpha(x,t) f^\beta(y,s) \succ = \delta(t-s) \int \frac{d^d p}{(2\pi)^d}\, e^{\imath p \cdot x} \left[\delta^{\alpha\,\beta} - \frac{p^\alpha p^\beta}{p^2} \right] \check{F}(p)$$

with

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with *power law spectrum* $d = 2$

$$\check{F}(p) = \frac{g_1 \, \nu^3 \, h_1(p, M, m)}{p^{d-4+2\,\varepsilon}}$$

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with *power law spectrum* $d = 2$

$$\check{F}(p) = \frac{g_1\,\nu^3\,h_1(p, M, m)}{p^{d-4+2\,\varepsilon}} + g_2\,\nu^3\,p^2\,h_2(p, M, m)$$

RG prediction

Energy spectrum

$$\mathcal{E}(q) = \varepsilon^{1/3}\, g_1^{2/3}\, \nu^2\, q^{1-\frac{4\,\varepsilon}{3}}\, R\left[\varepsilon, \frac{m}{q}, \left(\frac{q_b}{q}\right)^{2-\frac{2\,\varepsilon}{3}}\right] \qquad (\star)$$

$$q_b \propto \left[\frac{\varepsilon}{\nu^3 \tau^3}\right]^{\frac{1}{6-2\varepsilon}}$$

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If R has a limit for $q \downarrow 0$ the scaling prediction of $(\star)$ coincides with the scale by scale balance prediction

$$d_v - d_t = -\frac{d_t}{2} - d_x\,(2-\varepsilon)$$

$$2\,d_v - d_x = -\frac{d_t}{2} - d_x\,(2-\varepsilon)$$

RG prediction: 3d case

The energy spectrum prediction

$$\mathcal{E}(q) \sim q^{1-\frac{4\,\varepsilon}{3}} \qquad \varepsilon \leq 2$$

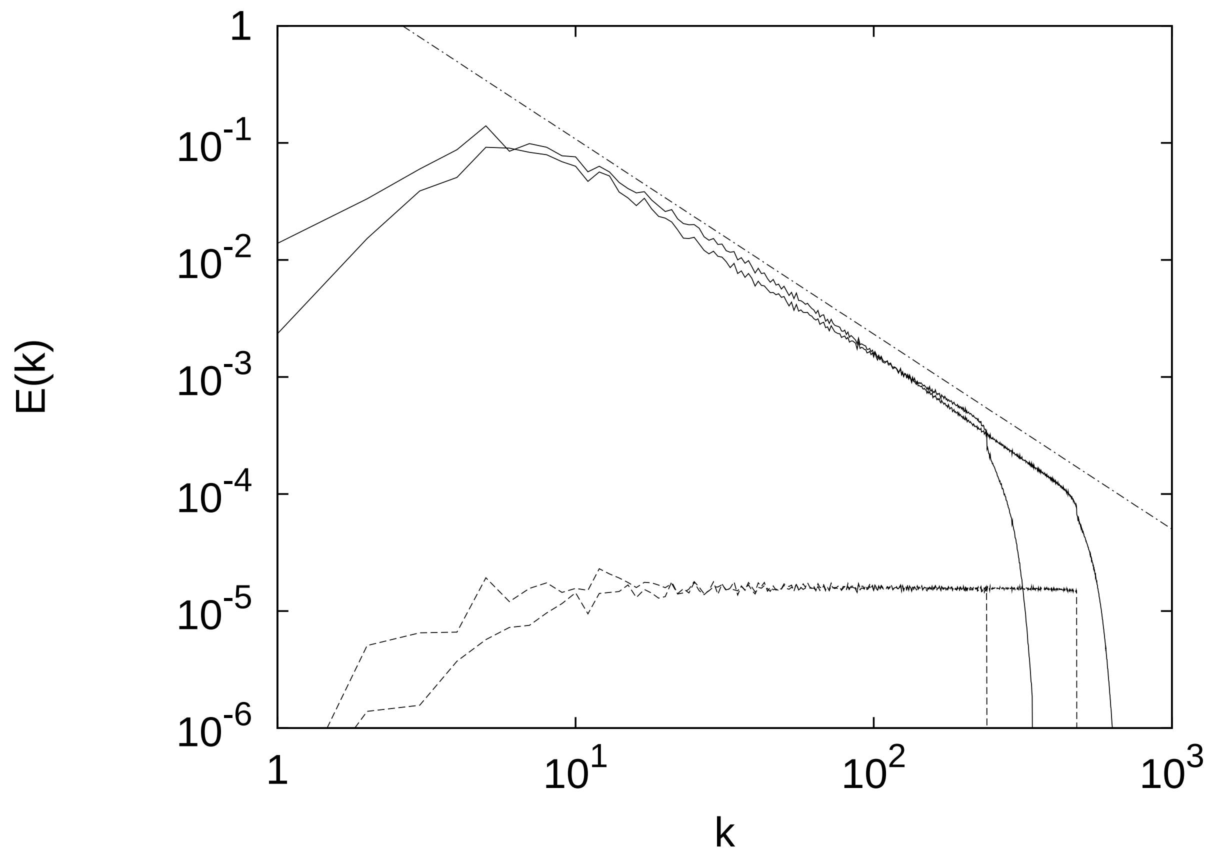
appears consistent with numerics in $3d$:

- A. Sain, Manu and R. Pandit, Phys. Rev. Lett. **81**, 4377 (1998).
- L. Biferale, A. Lanotte and F. Toschi, Phys. Rev. Lett. **92**, 094503 (2004).

Numerics

Numerical Energy Spectra

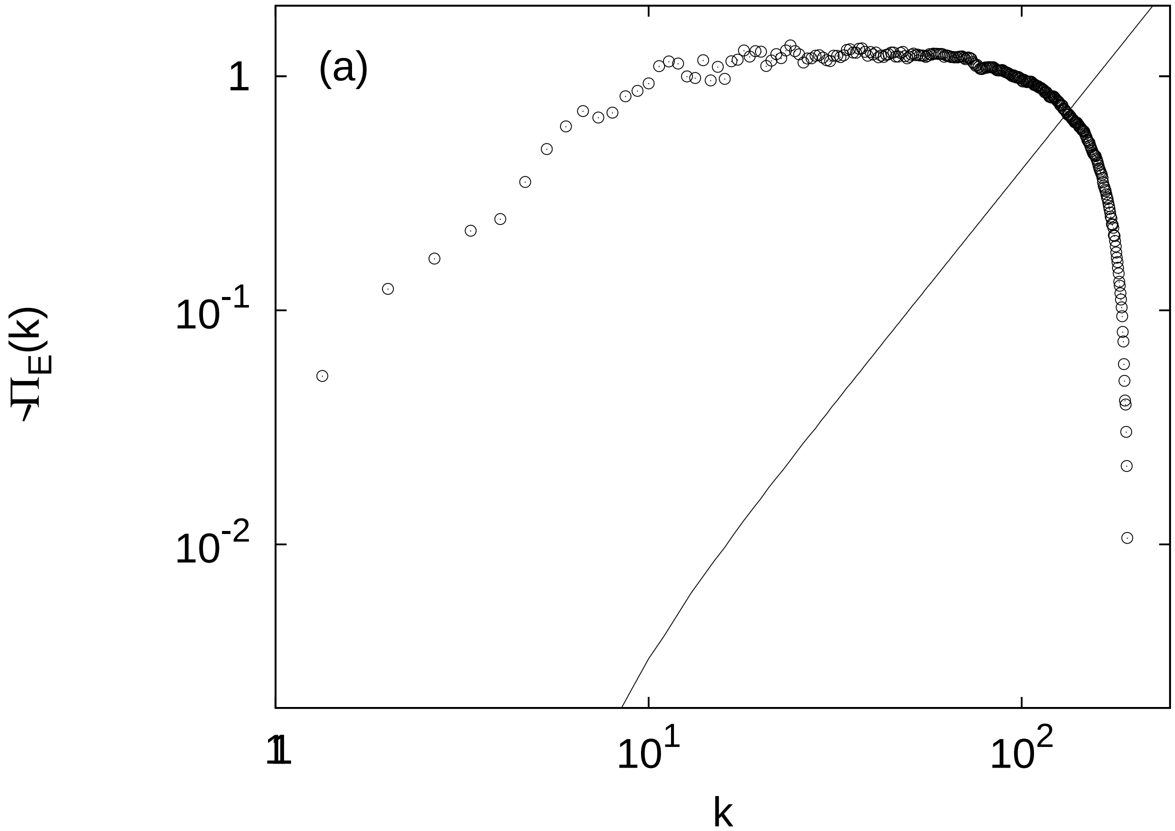
"Small scale forcing" $\varepsilon \leq 2$



Energy flux at $\varepsilon = 1$

Energy flux for $\epsilon = 1$ and integrated *energy input*

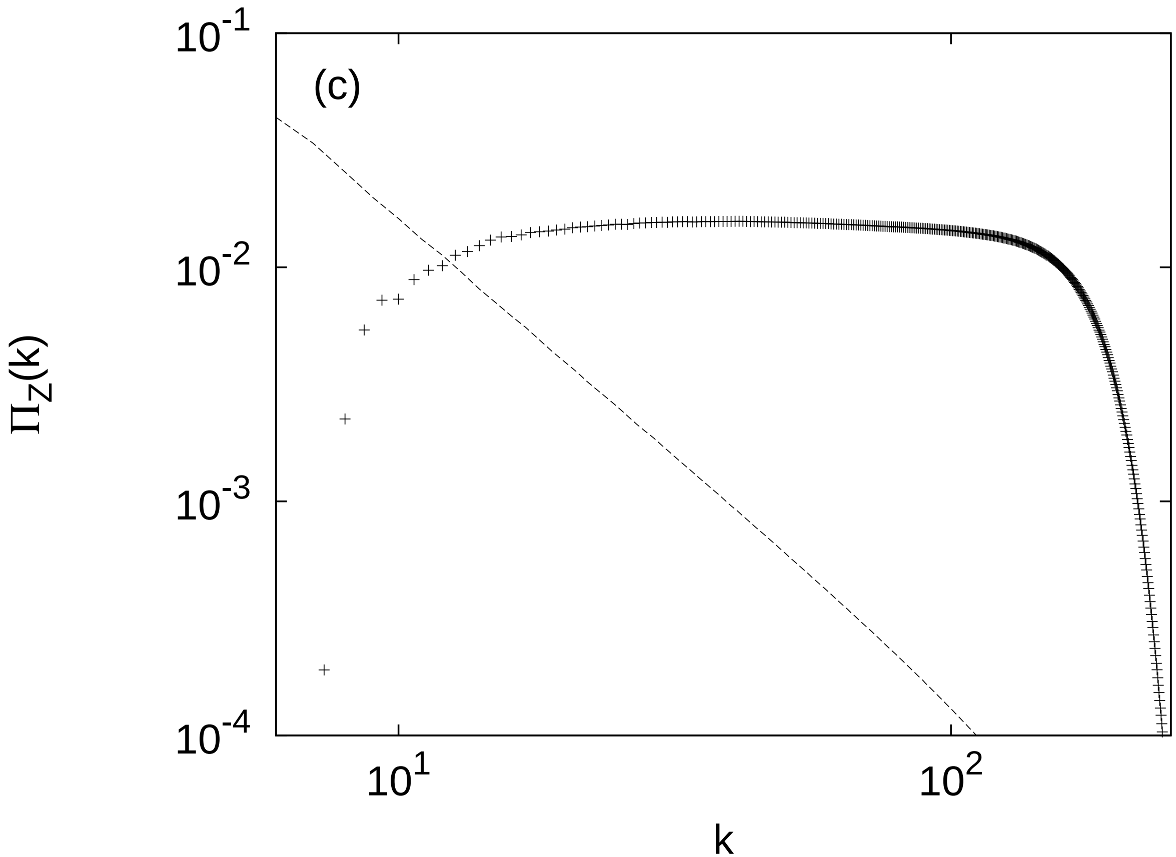
$$\int_{k \leq p} \frac{d^d k}{(2\pi)^d} \frac{1}{k^{d-4+2\varepsilon}} \bigg|_{d=2,\varepsilon=1} \sim p^2$$



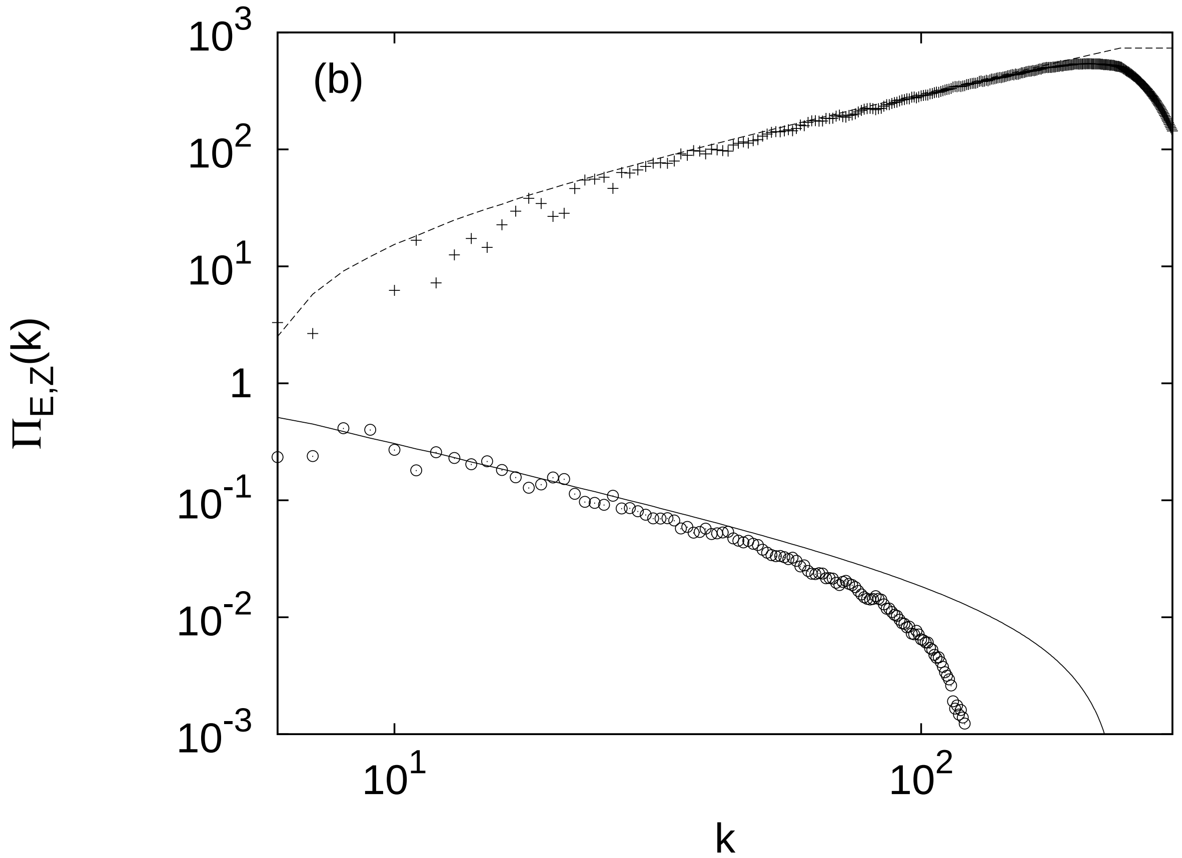
Enstrophy flux at $\varepsilon = 4$

Enstrophy flux for $\epsilon = 4$ and integrated *enstrophy input*

$$\int_{k \geq p} \frac{d^d k}{(2\pi)^d} \frac{k^2}{k^{d-4+2\varepsilon}}\bigg|_{d=2,\varepsilon=4} \sim p^{-2}$$

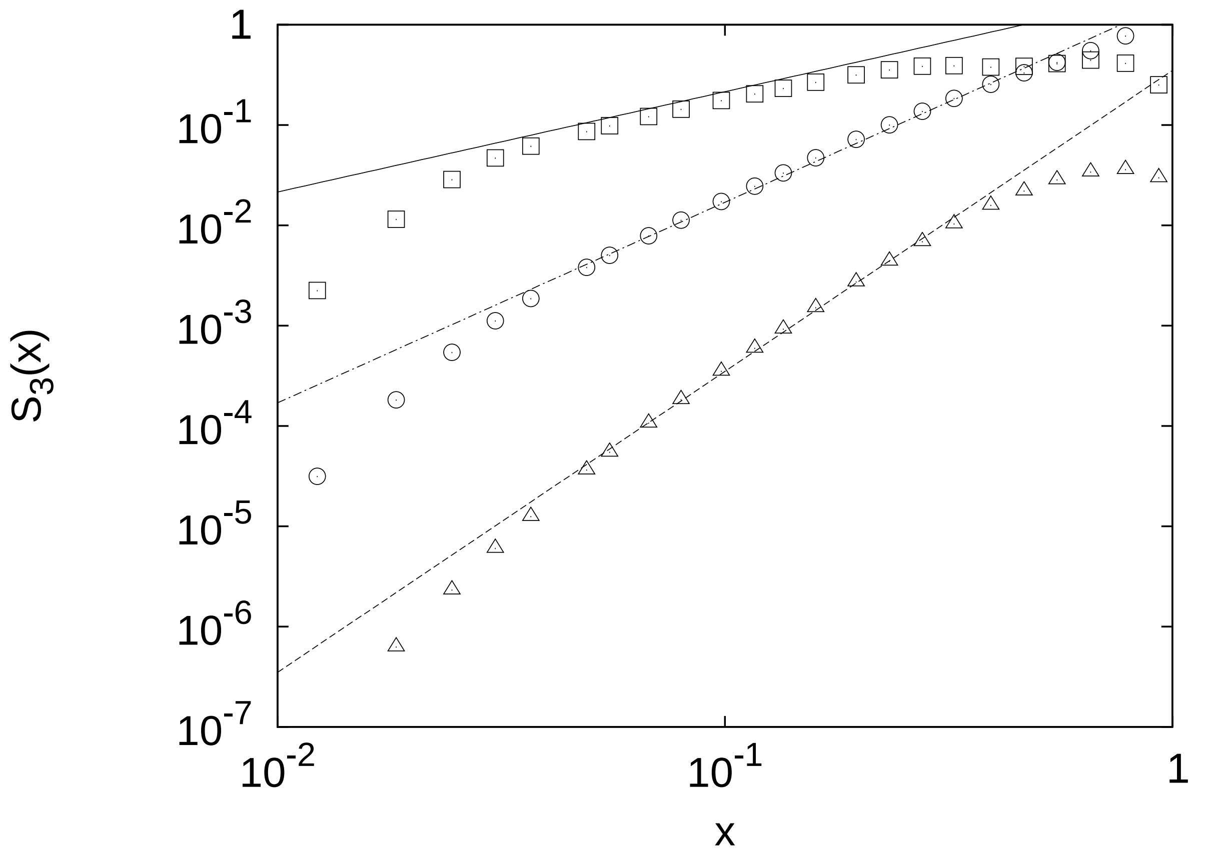


$\varepsilon = 2.5$



- circles: $\mathcal{E}$-flux line: integrated $\mathcal{E}$-input
- crosses: Ω-flux dashes: integrated Ω-input

Third order velocity structure functions



- $\varepsilon = 1$ (squares): $\delta v \sim x^{1/3}$ (inverse cascade).
- $\varepsilon = 2.5$ (circles): $\delta v \sim x^{-1+\frac{2\varepsilon}{3}}$ (dimensional).
- $\varepsilon = 4$ (triangles): $\delta v \sim x$ (direct cascade).

KHM for power law forcing: $0 \leq \varepsilon \leq 2$

$$-\frac{1}{2}\partial_\mu \prec \delta v^\mu(x)\delta \mathbf{v}^2(x) \succ =$$

$$\frac{\Sigma_2}{(2\,\pi)^2}\frac{F_0\,A(-4+2\,\varepsilon,2)}{(4-2\,\varepsilon)\,x^{4-2\,\varepsilon}} - \partial_t \prec v_\beta(x)\,v^\beta(0) \succ$$

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By hypotheses (i), (ii)

$$\partial_t \prec v_\beta(x)\,v^\beta(0) \succ = \frac{\Sigma_2}{(2\,\pi)^2}\frac{M^{4-2\,\varepsilon}\,\bar{F}_{4-2\,\varepsilon}}{4-2\,\varepsilon}$$

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The ratio between forcing and energy injection rate is

$$\frac{\partial_t \prec v_\beta(x)\,v^\beta(0) \succ}{F^\alpha{}_\alpha(x)} \propto (Mx)^{4-2\,\varepsilon} \gg 1$$

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$$\boxed{\prec \delta v_{||}^3 \succ \simeq M^{4-2\,\varepsilon}\,F_0\,c_1'\,x + \frac{c_2'\,F_0}{x^{3-2\,\varepsilon}} + \dots}$$

Conjecture

Unusual bifurcation of RG fixed points ?

R. Lipowski & M.E. Fisher, Phys. Rev. Lett. 57, 2411-2414 (1986).

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- Upper critical dimension $d_u = 3$.

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- Critical Phenomena for depinning of interfaces.
- Interfaces subject only to thermal fluctuations.
- Upper critical dimension $d_u = 3$.
- $d < d_u$: two lines of fixed point coalescing at $d = 3$ into a line of "drifting" fixed points:

$$\mathcal{R}(f^{\star}(l)) = \mathcal{R}(f^{\star}(l - \delta^{\star} l)) \qquad (d = 3)$$

Lipowski-Fisher bifurcation

$$\mathcal{R}\left\{f^{\star}(l)\right\} = f^{\star}(l) \qquad d < 3$$

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$$l \to l - \delta l$$

$$\mathcal{R}\left\{f^{\star}(l - \delta l)\right\} = f^{\star}\left(l - \frac{\delta l}{b^{\zeta}}\right)$$

$$\zeta = \frac{3 - d}{2}$$

Lipowski-Fisher bifurcation

$$\mathcal{R}\{f^\star(l)\} = f^\star(l) \qquad d < 3$$

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$$\mathcal{R}\{f^\star(l-\delta l)\} = f^\star\left(l - \frac{\delta l}{b^\zeta}\right)$$

$$\zeta = \frac{3-d}{2}$$

- $d < 3$: $\zeta > 0$ irrelevant perturbation.

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$$\zeta = \frac{3-d}{2}$$

- $d < 3$: $\zeta > 0$ irrelevant perturbation.
- $d = 3$: $\zeta = 0$ marginal perturbation.
- $d = 3$: stationary fixed points can be ruled out.

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- No obvious O.P.E. corrects the prediction.

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RG:

- No obvious O.P.E. corrects the prediction.
- Fixed point does not bifurcate from the Gaussian in the marginal limit.

THANK YOU!

KHM for power law forcing: $2 < \varepsilon < 3$

$$\partial_t \prec v_\beta(x)\, v^\beta(0) \succ -\frac{1}{2}\partial_\mu \prec \delta v^\mu(x)\delta \mathbf{v}^2(x) \succ \propto$$

$$\frac{m^{4-2\,\varepsilon}\,\bar{F}_{4-2\,\varepsilon}}{(2\,\varepsilon-4)} + \frac{m^{4-2\,\varepsilon}\,(m\,x)^2\,\bar{F}_{6-2\,\varepsilon}}{4\,(6-2\,\varepsilon)} - \frac{F_0\,A(2\,\varepsilon-4,2)\,x^{2\,\varepsilon-4}}{(2\,\varepsilon-4)} + \dots$$

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The injection rate is

$$\partial_t \prec v_\beta(x)\, v^\beta(0) \succ = \frac{\Sigma_2}{(2\,\pi)^2}\frac{m^{4-2\,\varepsilon}\, \bar{F}_{4-2\,\varepsilon}}{4 - 2\,\varepsilon}$$

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The KHM reduces to

$$-\frac{1}{2}\partial_\mu \prec \delta v^\mu(x)\delta \mathbf{v}^2(x) \succ =$$

$$\frac{\Sigma_2}{(2\,\pi)^2}\left\{\frac{m^{4-2\,\varepsilon}\,(m\,x)^2\,\bar{F}_{6-2\,\varepsilon}}{4\,(6-2\,\varepsilon)} - \frac{F_0\,A(2\,\varepsilon-4,2)\,x^{2\,\varepsilon-4}}{(2\,\varepsilon-4)} + \dots\right\}$$

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The structure function scales as

$$\prec \delta v_{||}^3 \succ \overset{(m\,x)\ll 1}{\simeq} F_0\, c_2' x^{2\,\varepsilon-3}\left\{1 + c_1'(m\,x)^{6-2\,\varepsilon} + \dots\right\}$$

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- The spectrum depends upon the large scale dissipation used in the numerics (Nam et al. 2000, Boffetta et al. 2002).

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$$\prec \delta v_{||}^3 \succ \overset{(m\,x) \ll 1}{\simeq} m^{6-2\,\varepsilon}\, F_0\, c_1' x^3 \left\{1 + c_2' (m\,x)^{2\,\varepsilon - 6} + \dots \right\}$$