2d inverse cascade: renormalization group theory versus numerical experiments

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Introduction

Outline

- Kolmogorov-Kraichnan’s theory of 2-d turbulence

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ii Renormalization group theory of 2-d turbulence

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iii Numerical experiments

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ii. Renormalization group theory of 2-d turbulence
iii. Numerical experiments

full story: Phys. Rev. Lett. **99**, (2007), 144502.

Velocity and Vorticity in $2d$

The Navier Stokes equation

$$(\partial_t + v \cdot \partial) v^\alpha - \nu\, \partial^2 v^\alpha = -\partial^\alpha P + f^\alpha \,, \qquad \alpha = 1, 2$$

$$\partial \cdot v = 0$$

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in *two dimensions* transports the vorticity field

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Conservation of vorticity moments in the inviscid limit.

Kraichnan’s picture of 2d turbulence

Energy and Enstrophy

Total Energy and Enstrophy equations

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- Energy dissipation vanishes as $\nu \downarrow 0$.
- Energy conservation $\Longrightarrow$ energy transfer towards small wavenumbers.
- Enstrophy dissipation at large wavenumbers.

Kàrmàn-Howarth-Monin equation

Gaussian, time short-correlated translational invariant forcing

$$\prec f^{\alpha}(x,t) f_{\alpha}(y,s) \succ = \delta(t-s)\, F(x-y, m)$$

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the Kàrmàn-Howarth-Monin (KHM) equation (equal times) is

$$\frac{1}{2} \prec (\partial \cdot \delta v)(x) \delta v(x)^2 \succ =$$

$$= \partial_t \prec v(x) \cdot v(0) \succ - F(x,m) - 2\,\nu \prec (\partial_\alpha v)(x)(\partial^\alpha v)(0) \succ$$

Hypotheses encoding Kraichnan's theory:

- velocity correlations are smooth at finite viscosity and exist in the inviscid limit even at coinciding points:

$$\lim_{x \to 0} \partial_t \prec v(x) \cdot v(0) \succ = I_{\mathcal{E}}$$

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iii No energy dissipative anomalies occur:

$$\left\{ \lim_{\nu\downarrow 0} \lim_{x\downarrow 0} - \lim_{x\downarrow 0} \lim_{\nu\downarrow 0} \right\} \nu \prec \partial_{\alpha} v^{\beta}(x,t) \partial^{\alpha} v_{\beta}(0,t) \succ = 0$$

KHM equation and "mean field" scaling

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Inverse Energy Cascade

$$\prec \delta v_{||}^3 \succ = \prec \delta v_{||}\delta v_\perp^2 \succ \overset{mx\gg 1}{=} \frac{3\, I_\mathcal{E}\, x}{2} \quad \overset{\text{mean field}}{\Rightarrow} \quad \delta v \sim x^{1/3}$$

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Direct Enstrophy Cascade

$$\prec \delta v_{||}^3 \succ = \prec \delta v_{||}\delta v_\perp^2 \succ \stackrel{\frac{x}{\ell}\ll mx \ll 1}{=} \frac{I_\Omega\, x^3}{8} \quad \stackrel{\text{mean field}}{\Rightarrow} \quad \delta v \sim x$$

Energy spectrum

$$\mathcal{E}(k) = \int \frac{d^d p}{(2\pi)^d}\delta(|k| - |p|) \int d^d x \, e^{\imath p\cdot x} \prec v(x) \cdot v(0) \succ$$

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G. Boffetta,

J. Fluid Mech.

589, 253 (2007).

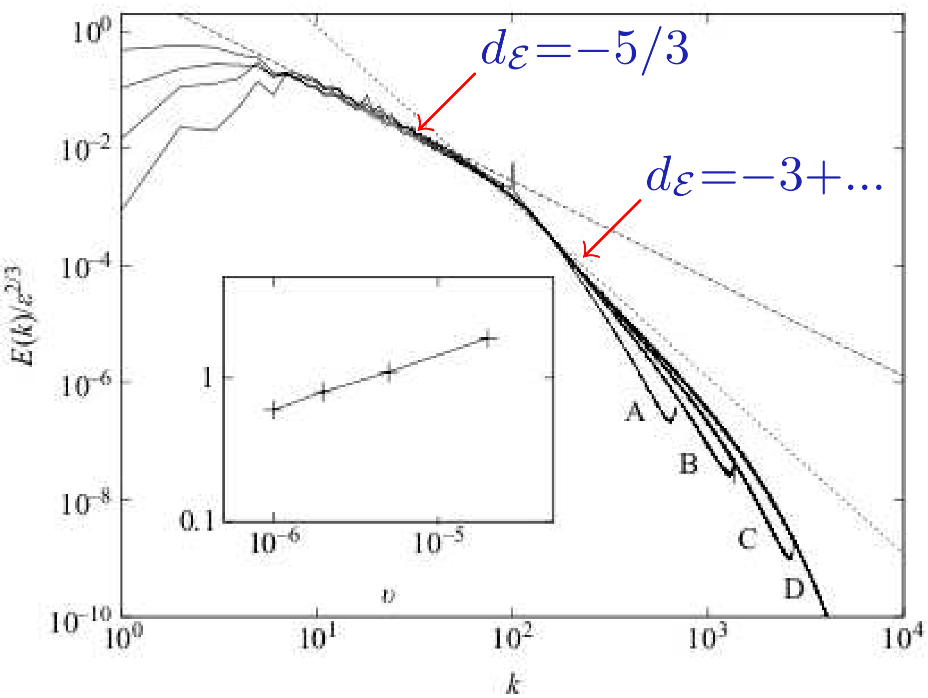

FIGURE 2. Energy spectra for the two simulations for the different resolutions (labels as in figure 1). Dashed and dotted lines represent the two predictions $Ck^{-5/3}$ with $C = 6$ and k^{-3} respectively. Inset: correction δ to the Kraichnan exponent -3 as a function of viscosity, computed by fitting the spectra with a power law $k^{-(3+\delta)}$ in the range $100 \leqslant k \leqslant 400$.

UV Renormalization Group analysis

P. Olla (1991),(1994)
Honkonen et al. (1998)

Power law forcing

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Ekman friction

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with *power law spectrum* $d = 2$

$$\check{F}(p) = \frac{g_1\, \nu^3\, h_1(p, M, m)}{p^{d-4+2\,\varepsilon}}$$

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- τ seems to play no role for RG !

RG prediction

Energy spectrum

$$\mathcal{E}(q) = \varepsilon^{1/3} g_1^{2/3} \nu^2 q^{1-\frac{4\varepsilon}{3}} R\left[\varepsilon, \frac{m}{q}, \left(\frac{q_b}{q}\right)^{2-\frac{2\varepsilon}{3}}\right] \qquad (\star)$$

$$q_b \propto \left[\frac{\varepsilon}{\nu^3 \tau^3}\right]^{\frac{1}{6-2\varepsilon}}$$

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If R has a limit for $q \downarrow 0$ the scaling prediction of $(\star)$ coincides with the scale by scale balance prediction

$$d_v - d_t = -\frac{d_t}{2} - d_x\,(2-\varepsilon)$$

$$2\,d_v - d_x = -\frac{d_t}{2} - d_x\,(2-\varepsilon)$$

RG prediction: 3d case

The energy spectrum prediction

$$\mathcal{E}(q) \sim q^{1-\frac{4\,\varepsilon}{3}} \qquad \varepsilon \leq 2$$

appears consistent with numerics in $3d$:

- A. Sain, Manu and R. Pandit, Phys. Rev. Lett. **81**, 4377 (1998).
- L. Biferale, A. Lanotte and F. Toschi, Phys. Rev. Lett. **92**, 094503 (2004).

Numerics

Model

We numerically integrate the *vorticity* equation

$$\omega = \epsilon_{\alpha\,\beta}\partial^{\alpha}v^{\beta}$$

$$\partial_t\omega + v\cdot\partial\omega = (-1)^{p+1}\nu\Delta^p\omega + \frac{(-1)^{q+1}}{\tau}\Delta^{-q}\omega + f_\omega$$

- Resolution up to 2048^2

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- $(-1)^{p+1} \nu \Delta^p \omega$: hyperviscosity.

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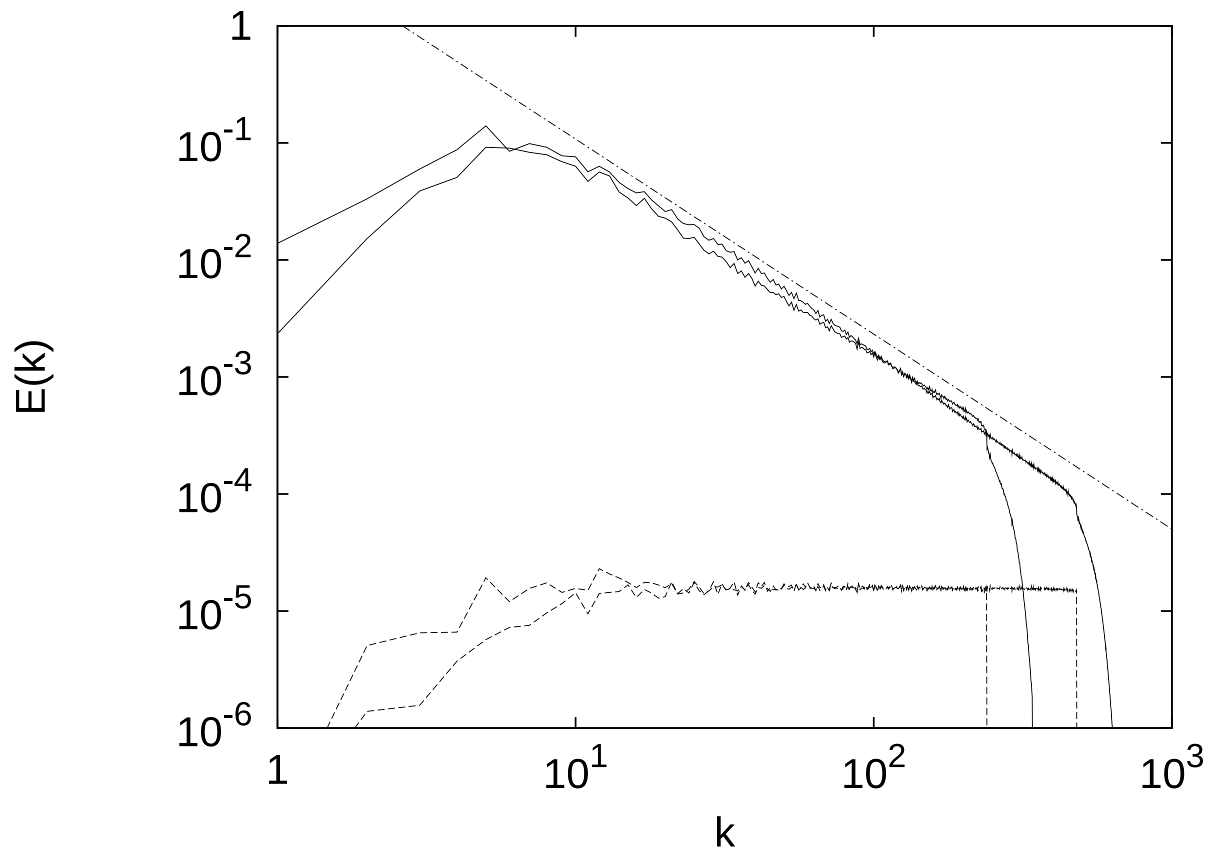
- $(p, q) = (4, 0)$ for $\varepsilon \leq 2$
- $(p, q) = (4, 1)$ for $\varepsilon \geq 3$ to inquire the inverse cascade.

Numerical Energy Spectra

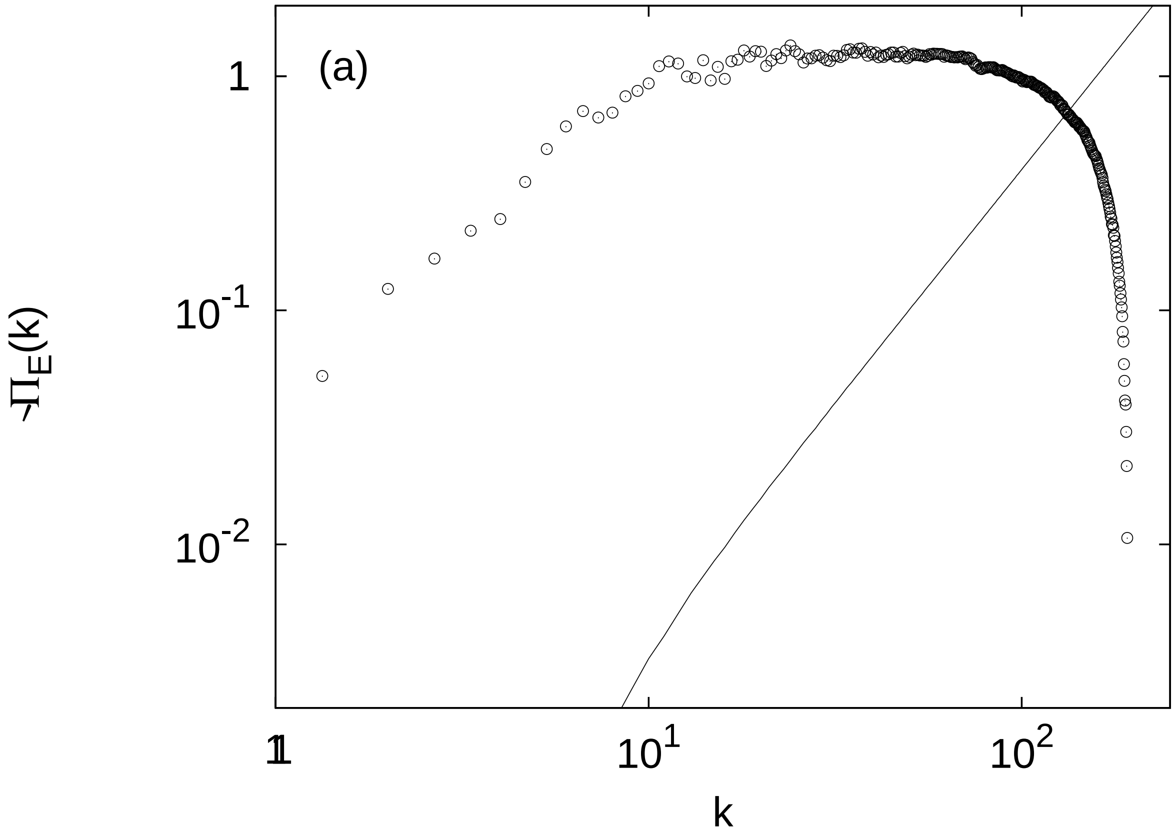
"Small scale forcing" $\varepsilon \leq 2$



Energy flux at $\varepsilon = 1$

Energy flux for $\epsilon = 1$ and integrated *energy input*

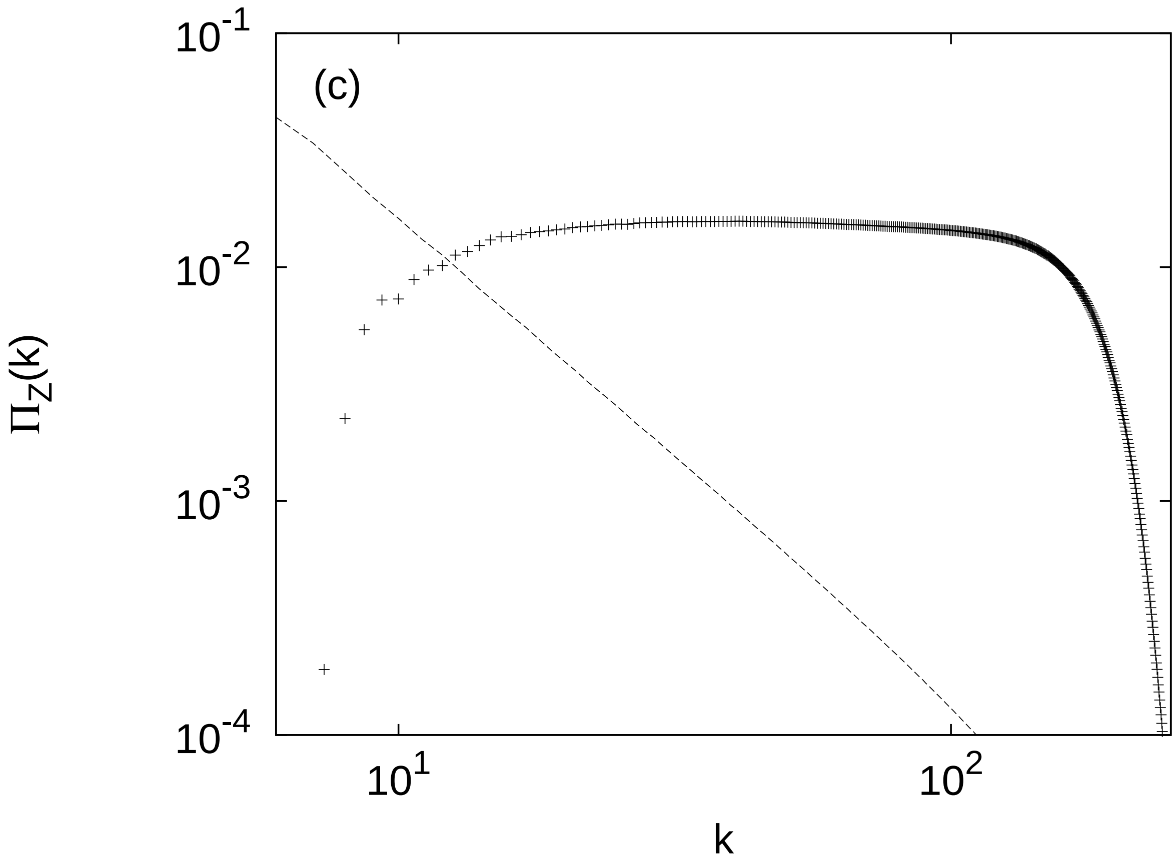
$$\int_{k\leq p} \frac{d^d k}{(2\pi)^d} \frac{1}{k^{d-4+2\varepsilon}}\bigg|_{d=2,\varepsilon=1} \sim p^2$$



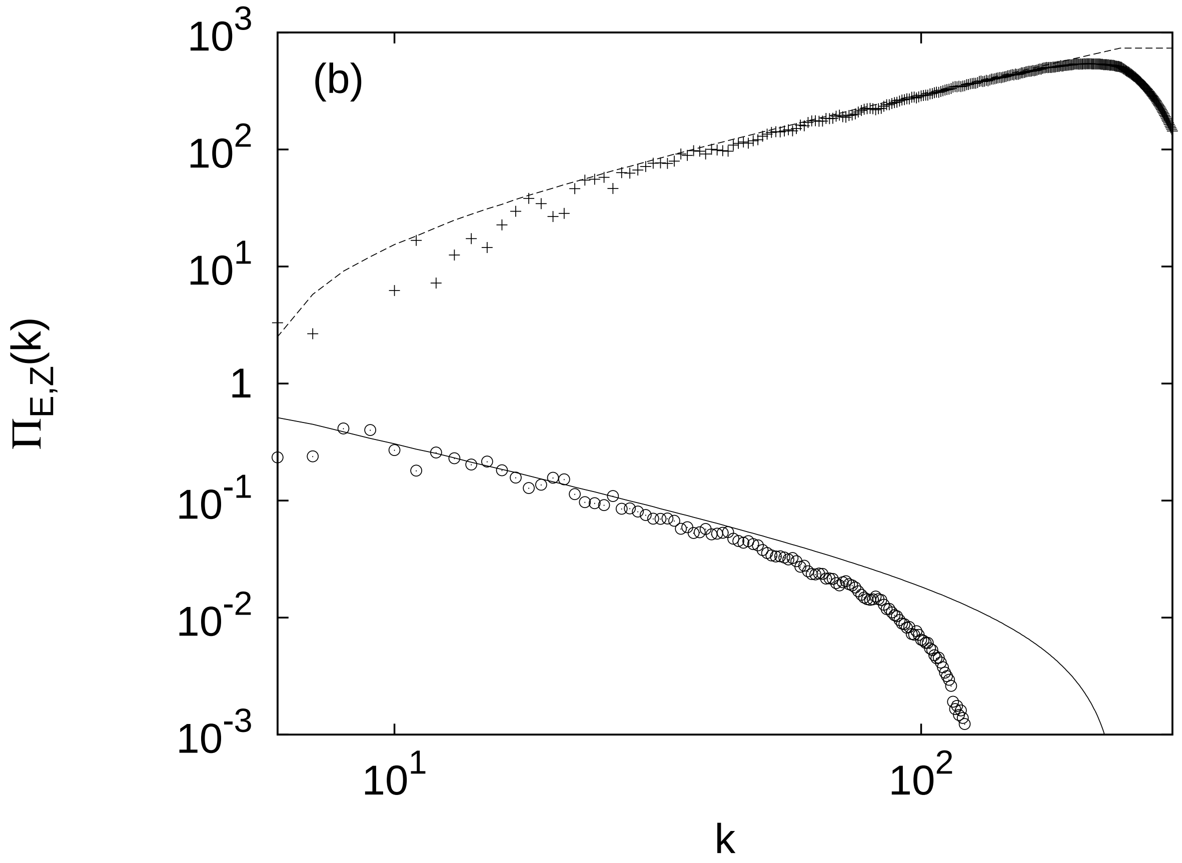
Enstrophy flux at $\varepsilon = 4$

Enstrophy flux for $\epsilon = 4$ and integrated *enstrophy input*

$$\int_{k \geq p} \frac{d^d k}{(2\pi)^d} \frac{k^2}{k^{d-4+2\varepsilon}} \Big|_{d=2,\varepsilon=4} \sim p^{-2}$$

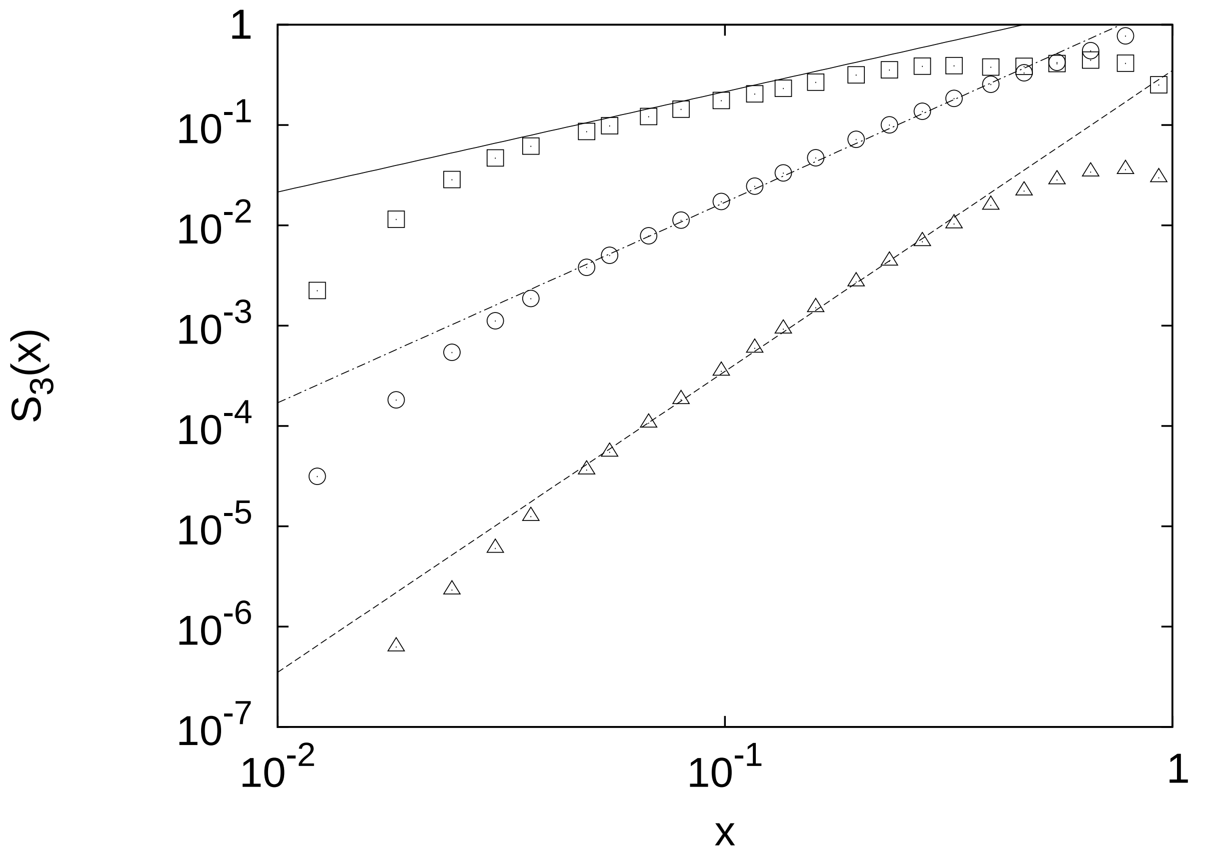


$\varepsilon = 2.5$



- circles: $\mathcal{E}$-flux line: integrated $\mathcal{E}$-input
- crosses: Ω-flux dashes: integrated Ω-input

Third order velocity structure functions



- $\varepsilon = 1$ (squares): $\delta v \sim x^{1/3}$ (inverse cascade).
- $\varepsilon = 2.5$ (circles): $\delta v \sim x^{-1+\frac{2\varepsilon}{3}}$ (dimensional).
- $\varepsilon = 4$ (triangles): $\delta v \sim x$ (direct cascade).

KHM for power law forcing: $0 \leq \varepsilon \leq 2$

$$-\frac{1}{2}\partial_\mu \prec \delta v^\mu(x)\delta \mathbf{v}^2(x) \succ =$$

$$\frac{\Sigma_2}{(2\,\pi)^2}\frac{F_0\,A(-4+2\,\varepsilon,2)}{(4-2\,\varepsilon)\,x^{4-2\,\varepsilon}} - \partial_t \prec v_\beta(x)\,v^\beta(0) \succ$$

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By hypotheses (i), (ii)

$$\partial_t \prec v_\beta(x)\,v^\beta(0) \succ = \frac{\Sigma_2}{(2\,\pi)^2}\frac{M^{4-2\,\varepsilon}\,\bar{F}_{4-2\,\varepsilon}}{4-2\,\varepsilon}$$

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The ratio between forcing and energy injection rate is

$$\frac{\partial_t \prec v_\beta(x)\,v^\beta(0) \succ}{F^\alpha{}_\alpha(x)} \propto (Mx)^{4-2\,\varepsilon} \gg 1$$

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$$\prec \delta v_{||}^3 \succ \simeq M^{4-2\,\varepsilon}\,F_0\,c_1'\,x + \frac{c_2'\,F_0}{x^{3-2\,\varepsilon}} + \dots$$

KHM for power law forcing: $2 < \varepsilon < 3$

$$\partial_t \prec v_\beta(x)\, v^\beta(0) \succ -\frac{1}{2}\partial_\mu \prec \delta v^\mu(x)\delta\mathbf{v}^2(x) \succ \propto$$

$$\frac{m^{4-2\,\varepsilon}\,\bar{F}_{4-2\,\varepsilon}}{(2\,\varepsilon-4)} + \frac{m^{4-2\,\varepsilon}\,(m\,x)^2\,\bar{F}_{6-2\,\varepsilon}}{4\,(6-2\,\varepsilon)} - \frac{F_0\,A(2\,\varepsilon-4,2)\,x^{2\,\varepsilon-4}}{(2\,\varepsilon-4)} + \dots$$

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The injection rate is

$$\partial_t \prec v_\beta(x)\, v^\beta(0) \succ = \frac{\Sigma_2}{(2\,\pi)^2}\frac{m^{4-2\,\varepsilon}\,\bar{F}_{4-2\,\varepsilon}}{4-2\,\varepsilon}$$

KHM for power law forcing: $2 < \varepsilon < 3$

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$$\frac{m^{4-2\,\varepsilon}\,\bar{F}_{4-2\,\varepsilon}}{(2\,\varepsilon-4)} + \frac{m^{4-2\,\varepsilon}\,(m\,x)^2\,\bar{F}_{6-2\,\varepsilon}}{4\,(6-2\,\varepsilon)} - \frac{F_0\,A(2\,\varepsilon-4,2)\,x^{2\,\varepsilon-4}}{(2\,\varepsilon-4)} + \dots$$

The KHM reduces to

$$-\frac{1}{2}\partial_\mu \prec \delta v^\mu(x)\delta\mathbf{v}^2(x) \succ =$$

$$\frac{\Sigma_2}{(2\,\pi)^2}\left\{\frac{m^{4-2\,\varepsilon}\,(m\,x)^2\,\bar{F}_{6-2\,\varepsilon}}{4\,(6-2\,\varepsilon)} - \frac{F_0\,A(2\,\varepsilon-4,2)\,x^{2\,\varepsilon-4}}{(2\,\varepsilon-4)} + \dots\right\}$$

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The structure function scales as

$$\prec \delta v_{||}^3 \succ \overset{(m\,x)\ll 1}{\simeq} F_0\,c_2' x^{2\,\varepsilon-3}\left\{1 + c_1'(m\,x)^{6-2\,\varepsilon} + \dots\right\}$$

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- The spectrum depends upon the large scale dissipation used in the numerics (Nam et al. 2000, Boffetta et al. 2002).

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Conjecture

Unusual bifurcation of RG fixed points ?

R. Lipowski & M.E. Fisher, Phys. Rev. Lett. 57, 2411-2414 (1986).

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- Critical Phenomena for depinning of interfaces.
- Interfaces subject only to thermal fluctuations.
- Upper critical dimension $d_u = 3$.
- $d < d_u$: two lines of fixed point coalescing at $d = 3$ into a line of "drifting" fixed points:

$$\mathcal{R}\{f^\star(l)\} = \mathcal{R}(f^\star(l - \delta^\star l)) \qquad (d = 3)$$

Lipowski-Fisher bifurcation

$$\mathcal{R}\left\{f^{\star}(l)\right\} = f^{\star}(l) \qquad d < 3$$

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$$\zeta = \frac{3 - d}{2}$$

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$l \to l - \delta l$

$$\mathcal{R}\{f^\star(l-\delta l)\} = f^\star\left(l - \frac{\delta l}{b^\zeta}\right)$$

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- $d < 3$: $\zeta > 0$ irrelevant perturbation.

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- $d = 3$: $\zeta = 0$ marginal perturbation.
- $d = 3$: stationary fixed points can be ruled out.

Conclusions

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RG:

- No obvious O.P.E. corrects the prediction.
- Fixed point does not bifurcate from the Gaussian in the marginal limit.