Generalized Schrödinger bridges in stochastic thermodynamics

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based on

Journal of Statistical Physics **191**, p. 117, (2024). arXiv: 2403.00679

in collaboration with

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November 19, 2024

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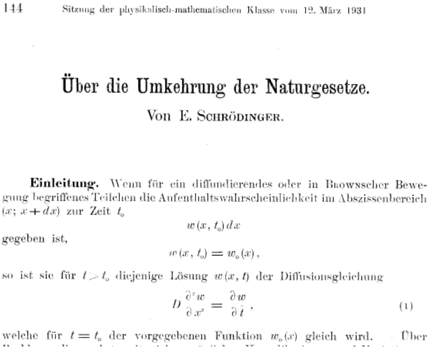
Standing on the shoulders of giants



144 Sitzung der physikalisch-mathematischen Klasse vom 12. März 1931

Über die Umkehrung der Naturgesetze.

Von E. Schrödinger.

Einleitung. Wenn für ein diffundierendes oder in Brownscher Bewegung begriffenes Teilchen die Aufenthaltswahrscheinlichkeit im Abszissenbereich $(x; x+dx)$ zur Zeit t_0

$$w(x, t_0)\, dx$$

gegeben ist,

$$w(x, t_0) = w_0(x),$$

so ist sie für $t > t_0$ diejenige Lösung $w(x, t)$ der Diffusionsgleichung

$$D \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t}, \qquad (1)$$

welche für $t = t_0$ der vorgegebenen Funktion $w_0(x)$ gleich wird. — Über

"if I have seen further [than others], it is by standing on the shoulders of giants. "

1675 Letter
from Sir Isaac Newton
to Robert Hooke

English translation Chetrite, Muratore-Ginanneschi, and Schwieger, *The European Physical Journal H* **46**, pp. 1–29, (2021). arXiv: 2105.12617

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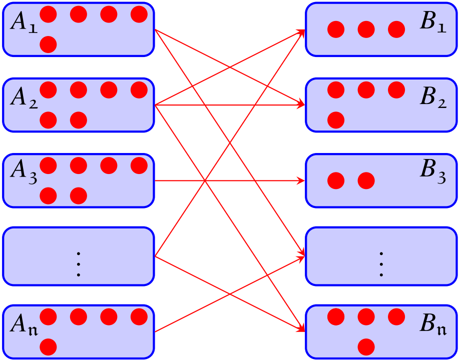
Schrödinger's particle migration model

- Two sets of $\mathfrak{n}$ boxes

  $\{A_i\}_{i=1}^{\mathfrak{n}}$ & $\{B_i\}_{i=1}^{\mathfrak{n}}$

- N particles initially randomly located in $\{A_i\}_{i=1}^{\mathfrak{n}}$
- Each *migration* is an independent event

  $g(j|i) = \Pr\left(\text{be in } B_j \,\middle|\, \text{been in } A_i\right)$



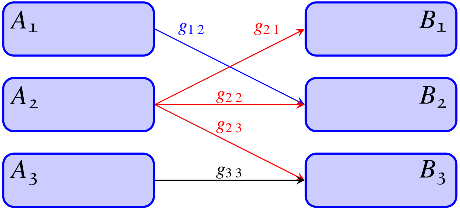
Schrödinger's bridge problem

H.1 : Assign initial marginal

$$\tilde{\mathscr{C}}_A = \boldsymbol{a}$$

H.2 : Assign final marginal

$$\tilde{\mathscr{C}}_B = \boldsymbol{b}$$



Q : Determine $\mathsf{K} = (k(i|j))$ "close to" $\mathsf{G} = (g(i|j))$ realizing the migration

Schrödinger's solution: minimize

$$\mathrm{D}_{KL}(\mathsf{K}\|\mathsf{G}) = \sum_{i\,j=1}^{n} k(j|i)\, w_{0}(i) \ln \frac{k(j|i)}{g(j|i)}$$

"The so–called irreversible laws of nature, if one interprets them statistically, do actually not privilege any time direction."

Modern formulations

Cost of the transition depends on the dynamics

Divergence from a reference process

$$\mathcal{P}, \mathcal{Q} = \text{path measures}$$

$$\mathrm{K}(\mathcal{P} \parallel \mathcal{Q}) = \mathrm{E}_{\mathcal{P}} \ln \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\mathcal{Q}}$$

Föllmer, “Time reversal on Wiener space” *Stochastic Processes, Mathematics and Physics* **vol. 1158**, pp. 119–129, (1986).
Dai Pra, *Applied Mathematics and Optimization* **23**, pp. 313–329, (1991).
Léonard, *Discrete and Continuous Dynamical Systems - Series A* **34**, pp. 1533–1574, (2014). arXiv: 1308.0215
Peyré and Cuturi, *Foundations and Trends in Machine Learning* **11**, pp. 355–607, (2019). arXiv: 1803.00567
Chen, Georgiou, and Pavon, *SIAM Review* **63**, pp. 249–313, (2021). ETC.

Mean entropy production (stochastic thermodynamics)

$$\mathcal{P}_{\mathcal{R}} = \text{time reversal + path reversal of } \mathcal{P}$$

$$\mathcal{E} = \mathrm{K}(\mathcal{P} \parallel \mathcal{P}_{\mathcal{R}}) = \mathrm{E}_{\mathcal{P}} \ln \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\mathcal{P}_{\mathcal{R}}}$$

Maes, Redig, and Moffaert, *Journal of Mathematical Physics* **41**, pp. 1528–1554, (2000).
Chétrite and Gawędzki, *Communications in Mathematical Physics* **282**, pp. 469–518, (2008). arXiv: 0707.2725
Peliti and Pigolotti, *Stochastic Thermodynamics*, Princeton University Press, (2020). ETC.

Mechanical Langevin-Kramers (underdamped) dynamics

Zwanzig, *Journal of Statistical Physics* **9**, pp. 215–220, (1973).
Cépas and Kurchan, *European Physical Journal B: Condensed Matter and Complex Systems* **2**, pp. 221–223, (1998). arXiv: cond-mat/9706296

Kinematics compatible with thermalization

$$d\boldsymbol{q}_t = \frac{\boldsymbol{p}_t}{m}dt - \frac{g\,\tau}{m}\,(\boldsymbol{\partial} U_t)(\boldsymbol{q}_t)dt + \sqrt{\frac{2\,g\,\tau}{m\,\beta}}\,d\boldsymbol{w}_t$$

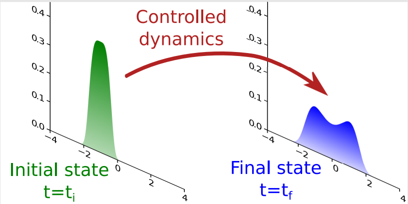
$$d\boldsymbol{p}_t = -\left(\frac{\boldsymbol{p}_t}{\tau} + (\boldsymbol{\partial} U_t)(\boldsymbol{q}_t)\right)dt + \sqrt{\frac{2\,m}{\tau\,\beta}}d\boldsymbol{\omega}_t$$

Physical motivation: open system in a bipartite environment

Cuccoli et al., *Physical Review E* **64**, p. 066124, (2001). (bipartite environment: Josephson junctions)

Generalized Schrödinger bridges

Find a mechanical potential steering the transition at minimum cost in a given time horizon



- Cost: divergence from a free diffusion

$$\mathrm{K}(\mathcal{P} \parallel \mathcal{Q}) = C\,\mathrm{E}_{\mathcal{P}} \int_{t_\iota}^{t_f} \mathrm{d}t\, \|(\boldsymbol{\partial} U_t)(\boldsymbol{q}_t)\|^2$$

$$C = \frac{\beta\,\tau\,(1+g)}{4\,m}$$

- Cost: entropy production – convexity problem!!

$$\mathcal{E} = \mathrm{E}_{\mathcal{P}} \ln \frac{\mathrm{p}_{t_\iota}(\boldsymbol{q}_{t_\iota}, \boldsymbol{p}_{t_\iota})}{\mathrm{f}_{t_f}(\boldsymbol{q}_{t_f}, \boldsymbol{p}_{t_f})} + \mathrm{E}_{\mathcal{P}} \int_{t_\iota}^{t_f} \mathrm{d}t \left(\frac{\beta\,\|\boldsymbol{p}_t\|^2}{m\,\tau} - \frac{d}{\tau} \right)$$

$$+ \frac{\beta\,g\,\tau}{m} \mathrm{E}_{\mathcal{P}} \int_{t_\iota}^{t_f} \mathrm{d}t \left(\|(\boldsymbol{\partial} U)(\boldsymbol{q}_t)\|^2 - \frac{(\boldsymbol{\partial}^2 U)(\boldsymbol{q}_t)}{\beta} \right)$$

Universal bound on the mean entropy production

Lower bound by Wasserstein 2-distance between the end PDFs

$$\mathcal{E}(underdamped) \geq \frac{m\,\beta}{(1+g)\,\tau}\frac{\mathrm{E}_{\mathcal{P}}\left\|\boldsymbol{q}_{t_f}-\boldsymbol{q}_{t_\iota}\right\|^2}{t_f-t_\iota} = \mathcal{E}(overdamped)$$

Aurell, Mejía-Monasterio, and Muratore-Ginanneschi, *Physical Review Letters* **106**, p. 250601, (2011). arXiv: 1012.2037
Aurell et al., *Journal of Statistical Physics* **147**, pp. 487–505, (2012). arXiv: 1201.3207
Chen, Georgiou, and Pavon, *Journal of Optimization Theory and Applications* **169**, pp. 671–691, (2016). arXiv: 1412.4430

} overdamped maps into optimal mass transport

Proofs:
Muratore-Ginanneschi, *Journal of Statistical Mechanics: Theory and Experiment* **2014**, P05013, (2014). arXiv: 1401.3394
Dechant and Sasa, *Physical Review E* **97**, p. 062101, (2018). arXiv: 1803.09447
Gawedzki, "Improved 2nd Law of Stochastic Thermodynamics for underdamped Langevin process",(2021)
Muratore-Ginanneschi and Peliti, *Journal of Statistical Mechanics: Theory and Experiment* **Aug 2023**, p. 083202, (2023). arXiv: 2302.08290

For a bipartite classical system exceeds that of a part

$$\mathrm{S}(\mathrm{Pr}(\xi,\eta)) = \mathrm{S}(\mathrm{Pr}(\xi)) + \mathrm{S}(\mathrm{Pr}(\eta|\xi)) \geq \mathrm{S}(\mathrm{Pr}(\xi))$$

First order conditions for optimality (stationary conditions)

Muratore-Ginanneschi, *Journal of Statistical Mechanics: Theory and Experiment* **2014**, P05013, (2014). arXiv: 1401.3394
Muratore-Ginanneschi and Schwieger, *Physical Review E* **90**, 060102(R), (2014). arXiv: 1408.5298

Mapping to a variational problem (model problem)

"Adjoint equation method"

$$\mathcal{A}[\mathcal{P}, V, U] = \mathrm{E}_{\mathcal{P}} \left(V_{t_\iota}(\boldsymbol{q}_{t_\iota}, \boldsymbol{p}_{t_\iota}) - V_{t_\iota}(\boldsymbol{q}_{t_f}, \boldsymbol{p}_{t_f}) \right) + \mathrm{E}_{\mathcal{P}} \int_{t_\iota}^{t_f} \mathrm{d}t \left(\frac{\beta\, \tau}{4\, m} \left\| (\boldsymbol{\partial} U_t)(\boldsymbol{q}_t) \right\|^2 + (\mathrm{D}\, V_t)(\boldsymbol{q}_t, \boldsymbol{p}_t) \right)$$

Mean forward derivative

Serrin, "Mathematical Principles of Classical Fluid Mechanics", Springer Science + Business Media, (1959).
Seliger and Whitham, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **305**, pp. 1–25, (1968).
Bismut, *SIAM Review* **20**, pp. 62–78, (1978).

equivalently (modulo technicalities)

Fokker-Planck control: $\boldsymbol{x} = [\boldsymbol{q}, \boldsymbol{p}]$

Fokker-Planck

$$\mathcal{A}[\mathrm{f}, U, V] = \int_{t_\iota}^{t_f} \mathrm{d}t \int_{\mathbb{R}^{2d}} \mathrm{d}^{2d}\boldsymbol{x} \left(\frac{\beta\, \tau}{4\, m} \left\| (\boldsymbol{\partial} U_t)(\boldsymbol{q}) \right\|^2 - V_t(\boldsymbol{x}) \left(\partial_t - \mathfrak{L}^{\dagger}_{\boldsymbol{x}} \right) \right) \mathrm{f}_t(\boldsymbol{x})$$

Lagrange multiplier

Breitenbach and Borzì, *Computational Optimization and Applications* **76**, 499–533, (2020).
Daudin, *Journal de Mathématiques Pures et Appliquées* **175**, 37–75, (2023). arXiv: 2109.14978

Mechanical potentials only depend upon “position”

Fokker–Planck evolution of the density:

$$(\partial_t - \mathfrak{L}_{\boldsymbol{x}}^{\dagger})\, \mathrm{f}_t(\boldsymbol{x}) = 0$$

Value function equation:

$$(\partial_t + \mathfrak{L}_{\boldsymbol{x}})\, V_t(\boldsymbol{x}) + \frac{\beta\,\tau}{4\,m} \left\| (\boldsymbol{\partial} U_t)(\boldsymbol{q}) \right\|^2 = 0$$

Condition specifying the optimal control potential ($g = 0$):

$$\Big((\boldsymbol{\partial} \ln \tilde{\mathrm{f}}_t)(\boldsymbol{q}) + \partial_{\boldsymbol{q}} \Big) \cdot \left(\int_{\mathbb{R}^d} \mathrm{d}^d\boldsymbol{p}\, \frac{\mathrm{f}_t(\boldsymbol{q},\boldsymbol{p})}{\tilde{\mathrm{f}}_t(\boldsymbol{q})}\, \partial_{\boldsymbol{p}} V_t(\boldsymbol{q},\boldsymbol{p}) - \frac{\beta\,\tau}{2\,m} (\boldsymbol{\partial} U_t)(\boldsymbol{q}) \right) = 0$$

$$\tilde{\mathrm{f}}_t(\boldsymbol{q}) = \int_{\mathbb{R}^d} \mathrm{d}^d\boldsymbol{p}\, \mathrm{f}_t(\boldsymbol{q},\boldsymbol{p})$$ (position marginal pdf)

Boundary conditions: kinematically compatible with equilibria

$$\mathrm{f}_{t_\iota}(\boldsymbol{x}) = \frac{e^{-\beta \left(\frac{\|\boldsymbol{p}\|^2}{2\,m} + U_\iota(\boldsymbol{q}) \right)}}{Z_\iota} \quad \& \quad \mathrm{f}_{t_f}(\boldsymbol{x}) = \frac{e^{-\beta \left(\frac{\|\boldsymbol{p}\|^2}{2\,m} + U_f(\boldsymbol{q}) \right)}}{Z_f}$$

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Takeaways

- Similar equations (more cumbersome!) hold for the entropy production.
- For the entropy production g-regularization guarantees convexity in the control.
- Convexity in the control $\Longrightarrow$ existence of stationary eqs.
- Q: uniqueness of solutions? (boundary not Cauchy problem!)
- Equations are non-local: average over momenta.
- Gaussian case: optimal potential equation reduces to a Lyapunov equation.
- Equations have the correct overdamped ($\varepsilon \searrow 0$) limit:

$$\lim_{g \searrow 0} \lim_{\varepsilon \searrow 0} \underline{\text{underdamped}} \text{ stationary eqs} = \underline{\text{overdamped}} \text{ stationary eqs}$$

Muratore-Ginanneschi and Schwieger, *Physical Review E* **90**, 060102(R), (2014). arXiv: 1408.5298

Analytical treatment: finite-time Poincaré-Lindstedt expansion around over-damped limit

Sanders, Baldovin, and Muratore-Ginanneschi, *Journal of Statistical Physics* **191**, p. 117, (2024). arXiv: 2403.00679
Sanders, Baldovin, and Muratore-Ginanneschi, *Eprint* , , (2024). arXiv: 2407.15678

Non dimensional variables identify a small parameter

$$t = \tau\, \mathtt{t} \quad \& \quad p = \sqrt{\frac{m}{\beta}}\, \mathtt{p} \quad \& \quad q = L\, \mathtt{q} \quad \& \quad U_t = \frac{1}{\beta}\, \mathtt{U}_t$$

$$\varepsilon = \sqrt{\frac{\tau^2}{m\, \beta\, L^2}}$$

Optimal control equations in the 1d case

$$\partial_{\mathtt{t}}\, \mathtt{f}_{\mathtt{t}} - \partial_{\mathtt{p}}\, (\mathtt{p} + \partial_{\mathtt{p}})\, \mathtt{f}_{\mathtt{t}} = \varepsilon \Big((\partial_{\mathtt{q}} \mathtt{U}) \partial_{\mathtt{p}} - \mathtt{p}\, \partial_{\mathtt{q}} \Big) \mathtt{f}_{\mathtt{t}}$$

$$\partial_{\mathtt{t}}\, \mathtt{V}_{\mathtt{t}} - \left(\mathtt{p}\, \partial_{\mathtt{p}} - \partial_{\mathtt{p}}^2 \right) \mathtt{V}_{\mathtt{t}} = \varepsilon \Big((\partial_{\mathtt{q}} \mathtt{U}) \partial_{\mathtt{p}} - \mathtt{p}\, \partial_{\mathtt{q}} \Big) \mathtt{V}_{\mathtt{t}} - \frac{\varepsilon^2}{4} \, \|(\partial \mathtt{U}_t)(\mathtt{q})\|$$

$$\int_{\mathbb{R}} \mathrm{d}\mathtt{p} \frac{\mathtt{f}_{\mathtt{t}}(\mathtt{p}, \mathtt{q})}{\tilde{\mathtt{f}}_{\mathtt{t}}(\mathtt{q})} \partial_{\mathtt{q}} \mathtt{V}_{\mathtt{t}}(\mathtt{p}, \mathtt{q}) = \frac{\varepsilon}{2} (\partial \mathtt{U}_t)(\mathtt{q})$$

Hierarchy in configuration space

Expansion in Hermite polynomials $\{\mathtt{H}_n\}_{n=0}^{\infty}$ unfolds the optimal control equations in an infinite hierarchy

$$\partial_{\mathtt{t}} f_{\mathtt{t}}^{(n)} = \mathscr{F}_n(f_{\mathtt{t}}^{(n-1)}, f_{\mathtt{t}}^{(n)}, f_{\mathtt{t}}^{(n+1)}, \partial_{\mathtt{q}} U_{\mathtt{t}})$$

$$\partial_{\mathtt{t}} v_{\mathtt{t}}^{(n)} = \mathscr{V}_n(v_{\mathtt{t}}^{(n-1)}, v_{\mathtt{t}}^{(n)}, v_{\mathtt{t}}^{(n+1)}, \partial_{\mathtt{q}} U_{\mathtt{t}})$$

$$\frac{\varepsilon}{2}(\partial \mathtt{U}_t)(\mathtt{q}) - \sum_{n=0}^{\infty}(n+1)!\,\frac{f_{\mathtt{t}}^{(n)}(\mathtt{q})\, v_{\mathtt{t}}^{(n+1)}(\mathtt{q})}{f_t^{(0)}(\mathtt{q})} = 0$$

$$\mathtt{f}_{\mathtt{t}}(\mathtt{p},\mathtt{q}) = \frac{e^{-\frac{\mathtt{p}^2}{2}}}{\sqrt{2\,\pi}} \sum_{n=0}^{\infty} f_{\mathtt{t}}^{(n)}(\mathtt{q})\,\mathtt{H}_n(\mathtt{p})$$

$$f_{\mathtt{t}_\iota}^{(n)}(\mathtt{q}) = \delta_{n,0}\,\frac{e^{-\mathtt{U}_\iota(\mathtt{q})}}{Z_\iota}$$

$$f_{\mathtt{t}_f}^{(n)}(\mathtt{q}) = \delta_{n,0}\,\frac{e^{-\mathtt{U}_f(\mathtt{q})}}{Z_f}$$

$$\mathtt{V}_t(\mathtt{p},\mathtt{q}) = \sum_{n=0}^{\infty} v_t^{(n)}(\mathtt{q})\mathtt{H}_n(\mathtt{p})$$

no b.c.

Poincaré-Lindstedt method
Wycoff and Bálazs, Physica A **146**, pp 175-200, (1987)

Slow times, e.g. $\varepsilon^2 \mathtt{t}$, treated as independent variables

$$f^{(n)}_{\mathtt{t}}(\mathtt{q}) = e^{-n\,\mathtt{t}} f^{(n:0)}_{\varepsilon^2\mathtt{t},\dots}(\mathtt{q}) + \sum_{k=1}^{\infty} \varepsilon^k f^{(n:k)}_{\mathtt{t},\varepsilon^2\mathtt{t},\dots}(\mathtt{q})$$

$$v^{(n)}_{\mathtt{t}}(\mathtt{q}) = e^{-n\,(t_f-\mathtt{t})} v^{(n:k)}_{\varepsilon^2\mathtt{t},\dots}(\mathtt{q}) + \sum_{k=1}^{\infty} \varepsilon^k\, v^{(n:k)}_{\mathtt{t},\varepsilon^2\mathtt{t},\dots}(\mathtt{q})$$

$$U_{\mathtt{t}}(\mathtt{q}) = U^{(k)}_{\varepsilon^2 t}(\mathtt{q}) + \sum_{k=1}^{\infty} \varepsilon^k\, U^{(k)}_{\mathtt{t},\varepsilon^2\mathtt{t},\dots}(\mathtt{q})$$

Cancellation of saecular terms (no need for slow variable $\varepsilon \mathtt{t}$)

$$f^{(n:i)}_{t_\iota,\varepsilon^2\mathtt{t},\dots}(\mathtt{q}) = f^{(n:i)}_{t_f,\varepsilon^2\mathtt{t},\dots}(\mathtt{q}) = 0 \qquad \forall\, n > 0 \quad \text{(boundary conditions)}$$

$$v^{(n:i)}_{t_\iota,\varepsilon^2\mathtt{t},\dots}(\mathtt{q}) = v^{(n:i)}_{t_f,\varepsilon^2\mathtt{t},\dots}(\mathtt{q}) \qquad \forall\, n \quad \text{(cost only when the potential varies)}$$

$$\mathcal{A}_\star = \sum_{i=0}^{\infty} \varepsilon^i \int_{\mathbb{R}^{2d}} \mathrm{d}q \left(v^{(0:i)}_{\varepsilon^2 t_\iota,\dots}(\mathtt{q})\, \mathtt{f}^{(0:0)}_{\iota}(\mathtt{q}) - v^{(0:i)}_{\varepsilon^2 t_f,\dots}(\mathtt{q})\, \mathtt{f}^{(0:0)}_{f}(\mathtt{q}) \right)$$

Perturbative intertwining of hierarchy elements

Leading order

$$f_{\mathtt{t}}^{(0)}(\mathtt{q}) = f_{\varepsilon^2\mathtt{t}}^{(0:0)}(\mathtt{q}) + \varepsilon^2 f_{\mathtt{t},\varepsilon^2\mathtt{t}}^{(0:2)}(\mathtt{q}) + O(\varepsilon^3)$$

$$f_{\mathtt{t}}^{(1)}(\mathtt{q}) = \varepsilon f_{\mathtt{t},\varepsilon^2\mathtt{t}}^{(1:1)}(\mathtt{q}) + O(\varepsilon^2)$$

$$f_{\mathtt{t}}^{(2)}(\mathtt{q}) = \varepsilon^2 f_{\mathtt{t},\varepsilon^2\mathtt{t}}^{(2:2)}(\mathtt{q}) + O(\varepsilon^3)$$

$$f_{\mathtt{t}}^{(n)}(\mathtt{q}) = o(\varepsilon^2) \quad \forall n \geq 3$$

$$v_{\mathtt{t}}^{(0)}(\mathtt{q}) = v_{\varepsilon^2\mathtt{t}}^{(0:0)}(\mathtt{q}) + O(\varepsilon^2)$$

$$v_{\mathtt{t}}^{(1)}(\mathtt{q}) = \varepsilon\, v_{\mathtt{t},\varepsilon^2\mathtt{t}}^{(1:1)}(\mathtt{q}) + O(\varepsilon^2)$$

$$v_{\mathtt{t}}^{(2)}(\mathtt{q}) = v_{\varepsilon^2\mathtt{t}}^{(2:0)}(\mathtt{q})\, e^{-2(t_f-\mathtt{t})} + O(\varepsilon)$$

$$v_{\mathtt{t}}^{(n)}(\mathtt{q}) = v_{\varepsilon^2\mathtt{t}}^{(n:0)}(\mathtt{q})\, e^{-n(t_f-\mathtt{t})} + O(\varepsilon)$$

Meaning of color code

Solutions of “cell” problem w.r.t. slow time $\varepsilon^2\mathtt{t}$ and $\mathtt{q}$ at order $O(\varepsilon^2)$.

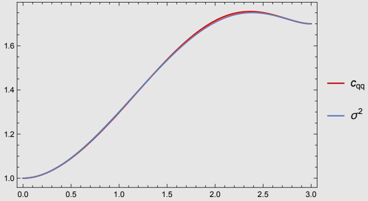
Solutions of differential problem only w.r.t. fast time $\mathtt{t}$ at order $O(\varepsilon)$ where $f_{\varepsilon^2\mathtt{t}}^{(0:0)}(q)$, $v_{\varepsilon^2\mathtt{t}}^{(0:0)}(q)$ and $v_{\varepsilon^2\mathtt{t}}^{(2:0)}(q)$ only appear as parameters.

Fixed by imposing that $f_{\varepsilon^2\mathtt{t}}^{(2:2)}$ satisfies the boundary conditions.

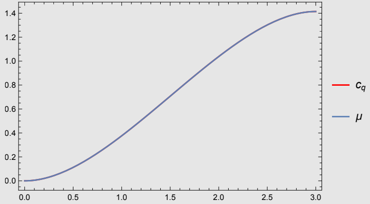
Enslaved or higher order quantities.

Gaussian case (numerics vs semi-analytical)

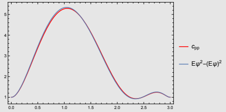
Cumulants exactly obey a finite set of ordinary differential equations



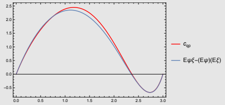
(a) position variance



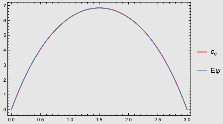
(b) position mean value



(c) momentum variance



(d) position momentum cross correlation



(e) momentum mean value

Figure: Horizon $t_f = 3$, $\varepsilon = 0.1$, red curves: non-perturbative numerics

Gaussian case

Second order cumulants and their co-states (non-dimensional)

$$\begin{cases} \dot{x}_t^{(1)} = 2\,\varepsilon\, x_t^{(2)} \\ \dot{x}_t^{(2)} = -x_t^{(2)} - \varepsilon \left(u_t^{(2)} x_t^{(1)} - x_t^{(3)} \right) \\ \dot{x}_t^{(3)} = 2 \left(1 - x_t^{(3)} - \varepsilon\, u_t^{(2)}\, x_t^{(2)} \right) \end{cases}$$

$$\begin{cases} \dot{y}_t^{(1)} = 2\,\varepsilon\, y_t^{(2)}\, u_t^{(2)} \\ \dot{y}_t^{(2)} = -\varepsilon\, y_t^{(1)} + y_t^{(2)} + \varepsilon\, y_t^{(3)}\, u_t^{(2)} \\ \dot{y}_t^{(3)} = -2\,\varepsilon\, y_s^{(2)} + 2\, y_s^{(3)} \end{cases}$$

B.C.: $\dot{x}_0^{(i)}$ & $\dot{x}_{t_f}^{(i)}$ $i = 1, 2, 3$ assigned

First order cumulants and their co-states (non-dimensional)

$$\begin{cases} \dot{x}_t^{(4)} = \varepsilon\, x_t^{(5)} \\ \dot{x}_t^{(5)} = -x_t^{(5)} + \varepsilon \left(u_t^{(1)} - u_t^{(2)}\, x_t^{(4)} \right) \end{cases}$$

$$\begin{cases} \dot{y}_t^{(4)} = y_t^{(2)} u_t^{(1)} + u_t^{(2)} y_t^{(5)} - \varepsilon\, u_t^{(2)} u_t^{(1)} \\ \dot{y}_t^{(5)} = y_t^{(5)} - \varepsilon\, y_t^{(4)} + \varepsilon\, y_t^{(3)} u_t^{(1)} \end{cases}$$

B.C.: $\dot{x}_0^{(i)}$ & $\dot{x}_{t_f}^{(i)}$ $i = 4, 5$ assigned

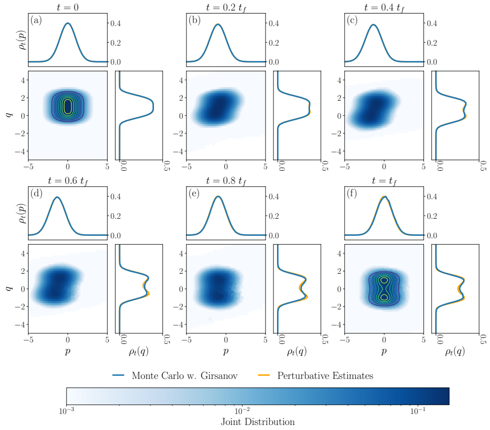
Numerical analysis

Sanders, Baldovin, and Muratore-Ginanneschi, *Eprint* , , (2024). arXiv: 2407.15678
Sanders and Muratore-Ginanneschi, *Eprint* , , (2024). arXiv: 2411.08518
& in progress

Self-consistence of perturbative drift

Sanders and Muratore-Ginanneschi, *Eprint* , , (2024). arXiv: 2411.08518

Landauer erasure at minimum divergence from free diffusion



Montecarlo method, we evolve $M = 10\,000$ trajectories from a set of 2601 equally spaced points from the interval $[-5, 5] \times [-5, 5]$. We use a time step size $h = 0.025$ and integrate over trajectories using an Euler-Maruyama discretization. We use $t_\iota = 0, t_f = 5, \beta = 25$ and $\tau = m = 1$. The potential in the initial condition is given by with $U_\iota(q) = \frac{1}{4}(q-1)^4$ and in the final condition with $U_f(q) = \frac{1}{4}(q^2-1)^2$.

Outlook

Exact numerical solution (in progress)

- Methods
  - Indirect methods: solution of the stationary equations (MonteCarlo +ML tools)
  - Direct methods: finding minimum of the cost functional (infinite optimization methods)
- Goal:
  - Underdamped half bridge.
  - Underdamped Landauer's erasure model in finite time.

THANKS, Julia & Marco

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