Quantum trajectory framework for general time-local master equations

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based on joint work with

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Evolution of an open quantum system state operator

Closed system: Liouville von-Neumann equation

$$\partial_t \boldsymbol{\omega}_t = -\imath \, [\mathbb{H} \, , \boldsymbol{\omega}_t]$$

deals with pure states or mixtures on the same footing

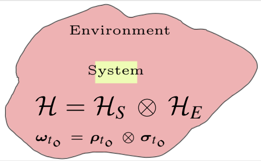
$$\boldsymbol{\omega}_t = \sum_i \wp_i \boldsymbol{\Psi}_t^{(i)} \boldsymbol{\Psi}_t^{(i)\dagger}$$

Open system: dynamical map (typical case)

$$\boldsymbol{\Phi}_{t\,t_\circ}(\boldsymbol{\rho}_{t_\circ}) = \mathrm{Tr}_{\mathcal{H}_E}\left(\mathrm{U}_{t\,t_\circ}\,\boldsymbol{\omega}_{t_\circ}\,\mathrm{U}^{\dagger}_{t\,t_\circ}\right)$$

$$\boldsymbol{\omega}_{t_\circ} = \boldsymbol{\rho}_{t_\circ} \otimes \boldsymbol{\sigma}_{t_\circ}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\rho}_t = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Phi}_{t\,t_\circ}(\boldsymbol{\rho}_{t_\circ}) = \text{Nakajima Zwanzig}$$



General time-local master equation

Vstovsky, *Physics Letters A*, (1973) & Grabert, Talkner, and Hänggi, *Zeitschrift für Physik B Condensed Matter*, (1977)
Wonderen and Lendi, *Journal of Statistical Physics*, (1995) & Andersson, Cresser, and Hall, *Journal of Modern Optics*, (2007)
Chruściński and Kossakowski, *Physical Review Letters*, (2010)

- finite dimensional Hilbert space
- $\boldsymbol{\rho}_s^{-1}$ exists in $[t_\circ, t_\mathrm{f}]$

Time-local canonical description

Hall et al., *Physical Review A*, (2014)

$$\partial_t \boldsymbol{\rho}_t = -\imath\, [\mathrm{H}_t\,, \boldsymbol{\rho}_t] + \sum_{\ell=1}^{d^2-1} \frac{w_{\ell;t}}{2} \left(\left[\mathrm{L}_{\ell;t}\,, \boldsymbol{\rho}_t\, \mathrm{L}_{\ell;t}^{\dagger}\right] + \left[\mathrm{L}_{\ell;t}\, \boldsymbol{\rho}_t\,, \mathrm{L}_{\ell;t}^{\dagger}\right] \right)$$

- $\mathrm{L}_{\ell;t}$ element of orthonormal frame.
- $w_{\ell;t}$ canonical rates: non-sign definite in general.
- $w_{\ell;t} \geq 0 \Leftrightarrow$ completely positive flow (Lindblad-Gorini-Kossakowski-Sudarshan).
- parametric dependence upon the instant of time when $\boldsymbol{\omega}_{t_\circ} = \boldsymbol{\rho}_{t_\circ} \otimes \boldsymbol{\sigma}_{t_\circ}$
- we assume $|w_{\ell;t}| < \infty$

Flow of the canonical master equation

$$\frac{\mathrm{d}\mathscr{B}_{t,s}}{\mathrm{d}t} = \mathrm{L}_{t-t_{\circ}}(\mathscr{B}_{t,s})$$
$$\mathscr{B}_{s,s} = \mathrm{Id}$$

$\mathscr{B}_{t,s} = \mathscr{B}_{t,v}\mathscr{B}_{v,s}$ for $s, v, t \in [t_{\circ}\ t_{\mathrm{f}}]$

- Action on operators

$$\boldsymbol{\rho}_t = \mathscr{B}_{t,s}(\boldsymbol{\rho}_s) = \sum_{a=1}^{\mathcal{N}^{(+)}} \mathrm{B}_{a;t\,s}^{(+)}\, \boldsymbol{\rho}_s\, \mathrm{B}_{a;t\,s}^{(+)\dagger} - \sum_{a=1}^{\mathcal{N}^{(-)}} \mathrm{B}_{a;t\,s}^{(-)}\, \boldsymbol{\rho}_s\, \mathrm{B}_{a;t\,s}^{(-)\dagger}$$

- completely positive **flow** if $\mathcal{N}^{(-)} = 0 \ \Leftrightarrow\ w_{\ell;t} \geq 0$
- completely positive evolution if $\boldsymbol{\rho}_s$ such that for $s\,t \in [s\,,t_{\mathrm{f}}]$ (more general)

$$\sum_{a=1}^{\mathcal{N}^{(-)}} \mathrm{B}_{a;t\,s}^{(-)}\, \boldsymbol{\rho}_s\, \mathrm{B}_{a;t\,s}^{(-)\dagger} = 0$$

- completely positive $\neq$ positivity preserving

Unraveling by the influence martingale

Donvil and Muratore-Ginanneschi, *Nature Communications*, (2022)

Completely positive case

Barchielli and Belavkin, *Journal of Physics A: Mathematical and General*, (1991), Dalibard, Castin, and Mølmer, *Physical Review Letters*, (1992) Carmichael, , (1993)

$$\boldsymbol{\rho}_t = \mathrm{E}\,\boldsymbol{\psi}_t\boldsymbol{\psi}_t^\dagger$$

- ψ_t squared norm stochastic process

Completely bounded case (including positivity preseving)

$$\boldsymbol{\rho}_t = \mathrm{E}\,\mu_t\boldsymbol{\psi}_t\boldsymbol{\psi}_t^\dagger$$

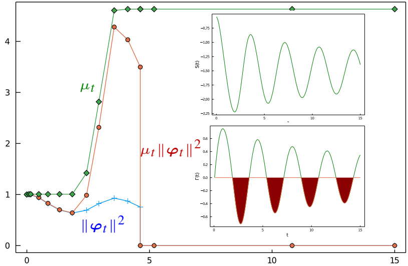
$$\mu_t^{(\pm)} = \max(0, \pm\mu_t) \qquad \Rightarrow \qquad \boldsymbol{\rho}_t = \mathrm{E}\left(\mu_t^{(+)}\psi_t\boldsymbol{\psi}_t^\dagger - \mu_t^{(-)}\psi_t\boldsymbol{\psi}_t^\dagger\right)$$

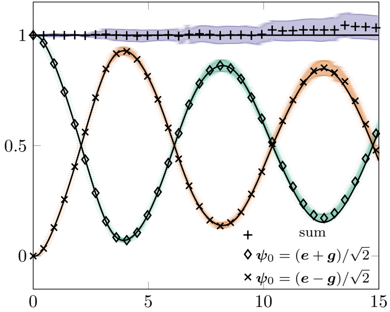
- ψ_t squared norm stochastic process
- μ_t mean preserving martingale

Photonic band gap John and Quang, *Physical Review A*, (1994)

Exact master equation, violates Kossakowski conditions

$$\dot{\boldsymbol{\rho}}_t = \frac{S_t}{2\imath}\left[\sigma_+\sigma_-\,,\boldsymbol{\rho}_t\right] + \Gamma_t\left(\left[\sigma_-\boldsymbol{\rho}_t\,,\sigma_+\right] + \left[\sigma_-\,,\boldsymbol{\rho}_t\sigma_+\right]\right)$$





Embedding $\mu_t = \frac{\lambda_t}{\mathrm{E}\,\lambda_t}$ with $\mathrm{E}\,\lambda_t$ **universal**!

add an ancilla: $|\lambda_t| \leq 1$

$$\varsigma_t = \frac{1_2 + \lambda_t\,\sigma_1}{2} \otimes \psi_t\psi_t^\dagger$$

it's a stochastic state operator!!!

Embedding completely positive evolution

$$\gamma_t = \mathrm{E}\,\frac{1_2 + \lambda_t\,\sigma_1}{2} \otimes \psi_t\psi_t^\dagger = \begin{bmatrix} \mathrm{E}\,\psi_t\psi_t^\dagger & \mathrm{E}\,\lambda_t\psi_t\psi_t^\dagger \\ \mathrm{E}\,\lambda_t\psi_t\psi_t^\dagger & \mathrm{E}\,\psi_t\psi_t^\dagger \end{bmatrix} = \begin{bmatrix} \tilde{\rho}_t & \varrho_t \\ \varrho_t & \tilde{\rho}_t \end{bmatrix}$$

- $\tilde{\rho}_t$, γ_t satisfy a completely positive master equation
- ϱ_t satisfies a trace non preserving completely bounded master equation.
- $|\,\mathrm{Tr}\,\varrho_t| \leq 1$

Recovery via embedding: protocol

Effect of time reversal

- completely positive flow $\Rightarrow$ completely bounded flow (inverse)

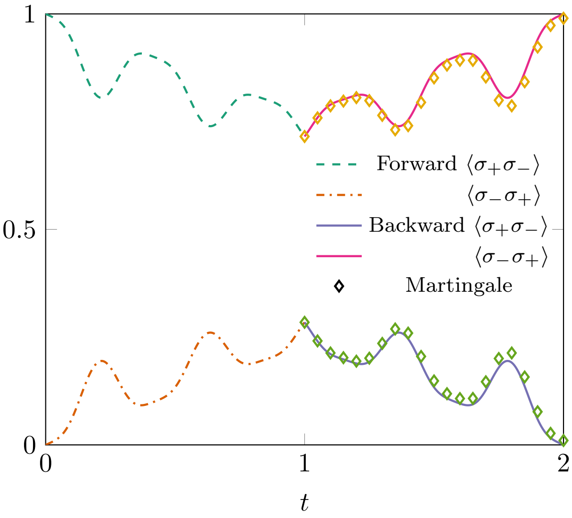
Embed a completely bounded flow into the off-diagonal corner of a completely positive flow

- System follows a completely positive evolution
- System is coupled to an ancillary qubit
- System+qubit follow the completely positive evolution with inverse flow on a off-diagonal corner
- Measurement of the ancilla:

$$\boldsymbol{\rho}_t \propto \mathrm{Tr}_1 \left(\sigma_1 \otimes 1_{\mathcal{H}} \, \mathrm{E} \left(\frac{1_2 + \lambda_t \sigma_1}{2} \otimes \psi_t \psi_t^\dagger \right) \right)$$

with universal proportionality factor.

Example driven qubit coupled to thermal environment



THANKS!

Technical slides

Unraveling pairs canonical master equations

$\boldsymbol{\rho}_t = \mathrm{E}\, \mu_t \boldsymbol{\psi}_t \boldsymbol{\psi}_t^\dagger$ solves the general master equation

$\tilde{\boldsymbol{\rho}}_t = \mathrm{E}\, \boldsymbol{\psi}_t \boldsymbol{\psi}_t^\dagger$ solves a paired completely positive master equation

Dynamics on the Bloch hyper-sphere

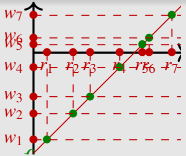
$$\mathrm{d}\,\boldsymbol{\psi}_t = \mathrm{d}t\,\boldsymbol{f}_t + \sum_{\ell=1}^{\mathscr{L}} \mathrm{d}\nu_{\ell,t} \left(\frac{\mathrm{L}_\ell\, \boldsymbol{\psi}_t}{\|\mathrm{L}_\ell\, \boldsymbol{\psi}_t\|} - \boldsymbol{\psi}_t \right)$$

$$\boldsymbol{f}_t = -\imath\, \mathrm{H}\, \boldsymbol{\psi}_t - \sum_{\ell=1}^{\mathscr{L}} r_{\ell,t} \frac{\mathrm{L}_\ell^\dagger\, \mathrm{L}_\ell\, \boldsymbol{\psi}_t - \|\mathrm{L}_\ell\, \boldsymbol{\psi}_t\|^2\, \boldsymbol{\psi}_t}{2}$$

Unraveling conditions: $w_{\ell,t} = r_{\ell,t} - c_t$

$$\mathrm{d}\mu_t = \mu_t \sum_{\ell=1}^{\mathscr{L}} \left(\frac{w_{\ell,t}}{r_{\ell,t}} - 1 \right) \left(\mathrm{d}\nu_{\ell,t} - \mathrm{E}\left(\mathrm{d}\nu_{\ell,t} \middle| \boldsymbol{\psi}_t, \bar{\boldsymbol{\psi}}_t \right) \right)$$

$$\mathrm{E}\left(d\nu_{\ell,t} \middle| \boldsymbol{\psi}_t, \bar{\boldsymbol{\psi}}_t \right) = r_{\ell,t}\, \|\mathrm{L}_\ell\, \boldsymbol{\psi}_t\|^2\, \mathrm{d}t$$



Optimization of the unraveling

One parameter family of unravelings: $c_t > \min_\ell \max(0, -w_{\ell,t})$

$$\min_{c_t} \operatorname{Tr}(\boldsymbol{\rho}_t - \tilde{\boldsymbol{\rho}}_t)^2 \leq \min_{c_t} \left(\mathrm{E}\, \mu_t^2 - 1\right)$$

- Universal (state independent) lowest upper bound attained for

$$c_t^\star = 2 \min_\ell \max(0, -w_{\ell,t})$$

- consequence on the influence martingale:

$$\mu_t \propto \lambda_t = \text{pure jump process}$$
$$|\lambda_t| \leq 1$$