Turbulent passive advection: shapes, geometry and multiscaling in the inertial and in the large scale decay ranges.

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Brisbane, Aug. 2008

Part I

Turbulence

Navier–Stokes and passive advection equations

- Navier–Stokes

$$(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}})\mathbf{v} - \nu\, \partial_{\mathbf{x}}^2 \mathbf{v} = \mathbf{f}_v - \frac{1}{\rho}\partial_{\mathbf{x}} P$$

$$\partial_{\mathbf{x}} \cdot \mathbf{v} = 0$$

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- passive advection

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- Reynolds, Prandtl & Péclet numbers

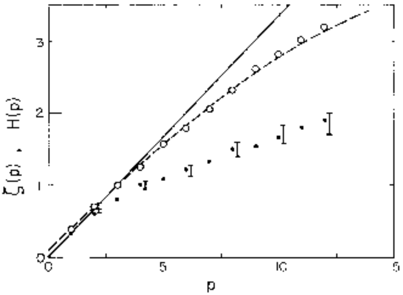
$$\mathrm{Re}_L := \frac{L\,V}{\nu} \qquad \mathrm{Pr} := \frac{\nu}{\kappa}$$

$$\mathrm{Pe}_L := \mathrm{Re}_L\,\mathrm{Pr}$$

Intermittency and Multiscaling ?

$$\prec [\hat{\mathbf{r}} \cdot (\mathbf{v}(\mathbf{r},t) - \mathbf{v}(0,t))]^p \succ \sim r^{\zeta(p)} \qquad \text{(circles)}$$

$$\prec [(\theta(\mathbf{r},t) - \theta(0,t))]^p \succ \sim r^{H(p)} \qquad \text{(bars)}$$



M. H. Jensen, G. Paladin and A. Vulpiani Phys. Rev. **A** 45, 7214 - 7221 (1992).

K41 theory

- Stochastic δ-correlated forcing for the velocity field

$$\prec f^{\alpha}(\mathbf{x},t) f^{\beta}(\mathbf{y},s) \succ = \delta(t-s) F^{\alpha\beta}(\mathbf{x}-\mathbf{y}), \qquad d \geq 3$$

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- Kármán-Howarth-Monin equation $\delta \mathbf{v}(\mathbf{x},\mathbf{t}) := \mathbf{v}(\mathbf{x},t) - \mathbf{v}(0,t)$

$$\left(\partial_t - \nu\,\partial^2\right) \prec v^{\alpha}(\mathbf{x},t) v_{\alpha}(0,t) \succ$$
$$-\frac{1}{2}\partial_{\mu} \prec ||\delta v||^2(\mathbf{x},t)\,\delta v^{\mu}(\mathbf{x},t) \succ = F^{\alpha}_{\alpha}(\mathbf{x})$$

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- Kolmogorov's exact results

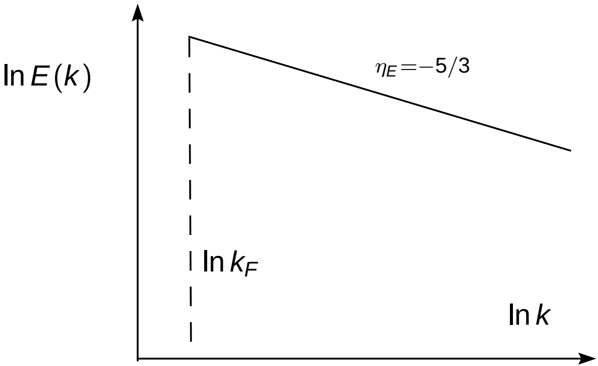
$$\lim_{\nu\downarrow 0}\lim_{x\downarrow 0} \nu \prec (\partial_{\beta} v^{\alpha})(\mathbf{x},t)(\partial^{\beta} v_{\alpha})(0,t) \succ = F^{\alpha}_{\alpha}(0) \quad \text{(dissip. anomaly)}$$

$$\lim_{x\downarrow 0}\lim_{\nu\downarrow 0} \prec [\hat{\mathbf{x}}\cdot\delta\mathbf{v}(\mathbf{x},\mathbf{t})]^3 \succ = -\frac{6\,F^{\alpha}_{\alpha}(0)\,x}{d\,(d+2)}\,, \qquad d > 2$$

K41 and the energy "cascade" picture

Energy spectrum

$$E(q) := \int \frac{d^d k}{(2\pi)^d}\,\delta(q-|k|)\int d^d x\, e^{-\imath k\cdot x}\, C_2{}^{\alpha}{}_{\alpha}(x)\,, \qquad d \geq 3$$



Part II

Kraichnan model

Obukhov (1948)

Kraichnan (1968), (1994)

Definition of the model

$$(\partial_t + \mathbf{v} \diamond \partial_{\mathbf{x}})\,\theta - \frac{\kappa}{2}\,\partial_{\mathbf{x}}^2\,\theta = f$$

$$\partial_{\mathbf{x}} \cdot \mathbf{v} = \prec f \succ = \prec \mathbf{v} \succ = 0$$

$$\prec v^{\alpha}(\mathbf{x}_1, t) v^{\beta}(\mathbf{x}_2, t_2) \succ = \delta(t_{12}) D^{\alpha\,\beta}(\mathbf{x}_{12}, m)$$

$$\prec f(\mathbf{x}_1, t_1) f(\mathbf{x}_2, t_2) \succ = \delta(t_{12}) F(m_f\, \mathbf{x}_{12})$$

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Ito representation of the SDE

$$(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}})\theta - \frac{\kappa_{eff.}}{2}\,\partial^2_{\mathbf{x}}\,\theta = f \qquad \text{Ito form}$$

$$\kappa_{eff.} = \kappa + \frac{D^{\alpha}_{\ \alpha}(0)}{d} \qquad \text{Taylor formula}$$

Consequences of time δ-correlation

- Galilean invariance of the statistics.

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- Closed hierarchy of equations for the equal time correlations

$$\left(\partial_t - \frac{\kappa}{2}\sum_{i=1}^{n}\partial^2_{\mathbf{x_i}} - \mathcal{M}_n\right) \prec \theta(\mathbf{x}_1,t)\ldots\theta(\mathbf{x}_n,t)\succ =$$

$$\frac{1}{2}\sum_{lk} F(\mathbf{x}_{lk}) \prec \theta(\mathbf{x}_1,t)\ldots\theta(\xcancel{\mathbf{x}_l,t})\ldots,\theta(\xcancel{\mathbf{x}_k,t})\ldots\theta(\mathbf{x}_n,t)\succ$$

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- Hypoelliptic "Hopf" operators

$$\mathcal{M}_n = \sum_{i\neq j=1}^{n} d^{\alpha\beta}(\mathbf{x}_{ij}) \frac{\partial}{\partial x_i^{\alpha}} \frac{\partial}{\partial x_j^{\beta}}$$

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- Structure tensor of the velocity field

$$d^{\alpha\,\beta}(\mathbf{x}) := D^{\alpha\,\beta}(0) - D^{\alpha\,\beta}(\mathbf{x})$$

Infinite inertial range of the advection

Re $\gg$ 1 & Pr $\ll$ 1

$$\lim_{m\downarrow 0} d^{\alpha\beta}(\mathbf{x}) = D_1 \, x^{\xi} \, \mathcal{I}^{\alpha\beta}(\hat{\mathbf{x}}, \xi) + O(m\,x)^2$$

$\mathcal{I}^{\alpha\beta}$ incompressibility projector

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$$\ell = \left(\frac{\kappa}{D_1}\right)^{\frac{1}{\xi}} \ll x \ll \mathrm{Min}\left(\frac{1}{m}, \frac{1}{m_f}\right)$$

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Decay: thermal equilibrium range ?

$$\frac{1}{m_f} \ll x \ll \frac{1}{m}$$

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- Existence of homogeneous translational invariant zero modes

$$\mathcal{M}_n Z(\mathbf{x}_1, \ldots, \mathbf{x}_n) = 0$$
$$Z(\lambda \mathbf{x}_1, \ldots, \lambda \mathbf{x}_n) = \lambda^{\zeta_n} Z(\mathbf{x}_1, \ldots, \mathbf{x}_n)$$

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- Zero modes yield inertial range asymptotics of the solutions as residues of the Mellin transform

$$\tilde{C}_{n;z}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \int_0^\infty \frac{dw}{w} \frac{\mathcal{C}(w\,\mathbf{x}_1, \dots, w\,\mathbf{x}_n; m_f, m)}{w^z}$$

Two point correlation

- Hopf equation

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- Mellin transform:

$$\int_0^\infty \frac{dw}{w}\frac{1}{w^z}\left[-d^{\alpha\beta}(wx)\frac{\partial}{\partial wx^\alpha}\frac{\partial}{\partial wx^\beta}C_2(wx,m_f)\right] = \int_0^\infty \frac{dw}{w}\frac{1}{w^z}F(wm_f x)$$

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$$d^{\alpha\beta}(x)\partial_\alpha\partial_\beta (m_f x)^{z+2-\xi}\,\tilde{C}(z+2-\xi) = \frac{(m_f x)^z\,\tilde{F}(z)}{-m_f^{2-\xi}}$$

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- Poles and zero modes

$$C_2(\mathbf{x}, m_f) \propto \frac{1}{m_f^{2-\xi}} \sum_{j\mathbf{l}} \int_{z^\star - \imath\infty}^{z^\star + \imath\infty} \frac{dz}{(2\pi\imath)} \frac{\tilde{F}_{j,\mathbf{l}}(z-2+\xi)\,(m_f x)^z\, Y_{j\,\mathbf{l}}(\hat{\mathbf{x}})}{(z-\zeta_2^{(s.s.)}(j))(z-\zeta_2^{(l.s.)}(j))}$$

Asymptotic analysis of the two point correlation

$C_2(\mathbf{x}, m_f) \propto$

$$m_f^{-2+\xi} \sum_{j\mathbf{l}} \int_{z^\star - \imath\infty}^{z^\star + \imath\infty} \frac{dz}{(2\pi\imath)} \frac{\tilde{F}_{j,\mathbf{l}}(z - 2 + \xi)\,(m_f x)^z\, Y_{j\,\mathbf{l}}(\hat{\mathbf{x}})}{(z - \zeta_2^{(s.s.)}(j))(z - \zeta_2^{(l.s.)}(j))}$$

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Canonical dimension

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- Small scales (inertial range)

$$\begin{aligned}\zeta_2^{(s.s.)}(j) &= \frac{2-d-\xi}{2}\left\{1-\sqrt{1+\frac{4j\,(d-2+j)\,(d+\xi-1)}{(d-1)\,(d+\xi-2)^2}}\right\} \\ &\overset{\xi\downarrow 0}{\longrightarrow} j+\frac{j\,(j-1)\,\xi}{(d-1)\,(d-2+2j)}+O(\xi^2)\end{aligned}$$

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$$\zeta_2^{(l.s.)}(j) = -\tfrac{d+\xi-2}{2}\left\{1 + \sqrt{(d+\xi-2)^2 + \tfrac{4j\,(d-2+j)\,(d+\xi-1)}{d-1}}\right\}$$

$$\stackrel{\xi\downarrow 0}{\rightarrow} (2-d-j)\left\{1 + \frac{(d+j-1)\xi}{(d-1)\,(d+2j-2)}\right\} + O(\xi^2)$$

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$$C_2(\mathbf{x}, m_f) \simeq x^{2-\xi}\,(m_f\, x)^{-d}\; \tilde{c}_0\, \check{F}(0)$$

$$+ x^{2-\xi}\,(m_f\, x)^{\zeta_2^{(l.s.)}(2) - \zeta_2(0)^{l.s.}}\; \tilde{c}_2\, \check{F}_2(0) + \dots$$

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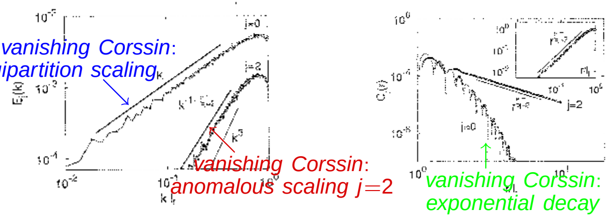
Passive advection by Navier–Stokes



A. Celani & A. Seminara, (2006).

Perturbative results (small ξ)

- Expansion of the Hopf operators

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- Expansion of the Hopf operators

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- Free theory in the translational invariant sector

$$\mathcal{M}_n^{(0)} = \frac{D^{\alpha}{}_{\alpha}(0,m)}{2\,d} \sum_{i=1}^{n} \partial_i^2$$

$$\mathcal{M}_n^{(0)}\,\mathcal{G}^{(0)}_{\mathfrak{d}_n,n} = 1 \qquad \mathfrak{d}_n := d\,(n-1)$$

admits a well-known explicit zero mode expansion

$$\mathcal{G}^{(0)}_{\mathfrak{d}_n,n}(\mathbf{r}-\mathbf{w}) = \sum_{j\,\mathbf{l}}^{\infty} H_{j,\mathbf{l}}(\mathbf{r})\,(\mathcal{K}\circ H_{j\,\mathbf{l}})(\mathbf{w}) + \mathbf{r} \Leftrightarrow \mathbf{w}$$

$$H_{j,\mathbf{l}}(\mathbf{r}) = \text{Harmonic polynomial}$$

$$(\mathcal{K}\circ H_{j\,\mathbf{l}})(\mathbf{r}) = \text{Kelvin transform of } H_{j,\mathbf{l}}(\mathbf{r}) = \frac{1}{r^{\mathfrak{d}_n-2}} H_{j,\mathbf{l}}\left(\frac{\mathbf{r}}{r^2}\right)$$

$$(\mathbf{r}\,,\mathbf{w}) = \text{Jacobi coordinates}$$

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  - $\forall\, \mathcal{M}_n\ \exists\,!$ *irreducible*
- Interaction at leading order

$$\mathcal{M}_n^{(1)} = \sum_{i \neq j}^{n} d_{(1)}^{\alpha\,\beta}(\mathbf{x}_{ij})\partial_{i\,\alpha}\partial_{j\,\beta}$$

$$d_{(1)}^{\alpha\,\beta}(\mathbf{x}) = D^{\alpha}{}_{\alpha}(0,1)\left\{\ln(m\,x)\delta^{\alpha\,\beta} - \frac{1}{d-1}\frac{x^\alpha\, x^\beta}{x^2}\right\} + O(m\,x)^2$$

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The symmetry goes down to

$$SO(\mathfrak{d}_n) \rightarrow \Sigma_n \times SO(d)$$

Zero modes are found by *diagonalisation* Gawędzki & Kupiainen (1995)

Inertial range anomalous scaling

$$S_{2n}(x) := \prec [\theta(\mathbf{x},t) - \theta(0,t)]^{2n} \succ \simeq \delta Z_{2n,ir} + O(mx)$$

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Inertial range scaling and U.V. singularities

- OPE: U.V. R.G. of the local field functionals

$$G_{2n} = [x^{\alpha}\partial_{\alpha}\theta(0,t)]^{2n}$$

determines the ρ_{2n}'s perturbatively for $\xi \downarrow 0$

Antonov et al (1998), (2001), Kupiainen and Muratore-Ginanneschi (2006) .

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- Inversion of limits Ansatz (OPE)

$$\lim_{|x|\downarrow 0}\frac{\mathcal{S}_{2n}(x; m, M)}{|x|^{2\,n}} \propto M^{n\xi}\left(\frac{M}{m}\right)^{\rho_{2n}}$$

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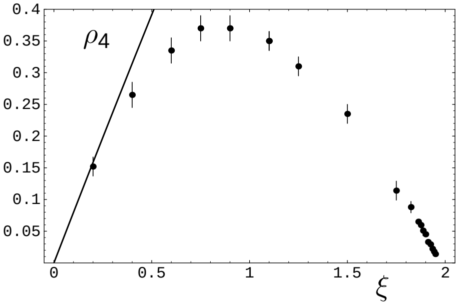
Self-symmetry breaking by the I.R. scale

$$S_{2n}(x) = \sum_{\sigma \in \mathcal{S}_{2n} / \mathcal{S}_n \times \mathcal{S}_2^{2n}} \sigma \int \prod_{j=1}^{n} dt_j \prec \prod_{i=1}^{n-1} F(x_{i\,i+1}(t_j)) \succ$$



Frisch Mazzino and Vergassola (1999), Mazzino and Muratore-Ginanneschi (2001)

Large scale zero modes

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- Four point by explicit diagonalisation Muratore-Ginanneschi (2008)

$$\sigma_{4,0}([4,0]) = -\xi \left\{ \frac{2\,(d+4)}{d+2} - 1 \right\} + O(\xi^2) \qquad \text{("dual" to irreducible)}$$

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- RG type analysis

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U.V. singularities and large scale decay of *zero modes*

- For

$$\frac{1}{m_f} \ll \frac{1}{m} \ll x$$

canonical scaling analysis yields

$$Z_{2n}(x, m\,x, M\,x) = \frac{g_{2n}(m_f x\,, m\,x, M\,x)}{x^{\mathfrak{d}_n - 2 + 2n}}$$

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- Operator product expansion

$$\lim_{m_f \downarrow 0} \tilde{s}_{2n}\left(\frac{m}{m_f}\,, \frac{M}{m_f}\right) \propto M^{n\xi}\left(\frac{M}{m}\right)^{\rho_{2n}}$$

yields a prediction for the scaling in m_f.

The probabilistic meaning of zero modes

Bernard, Gawędzki and Kupiainen (1996), Falkovich, Gawędzki and Vergassola (2001)

- The scale-invariant semigroup

$$P_n(\mathbf{X}, t \,|\, \mathbf{X}_0, t_0) = e^{(t-t_0)\mathcal{M}_n}(\mathbf{X}, \mathbf{X}_0)$$
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- is the transition probability density of the Markov process $\kappa \downarrow 0$ Le Jan and Ramon (1998)

$$d\,\mathbf{x}_t = \mathbf{v}(\mathbf{x}_t\,, t)\,dt$$
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- governs the evolution of Lagrangian averages

$$\prec f \succ_{(t|\mathbf{X}_0,t_0)} = \int dX_0\,f(\mathbf{X})\,P_n(\mathbf{X},t\,|\,\mathbf{X}_0,t_0)$$

Coherent structures preserved by the flow

- $f(\lambda \mathbf{X}) = \lambda^{\sigma} f(\mathbf{X}) \quad \Rightarrow \quad \prec f \succ_{(t|\mathbf{X}_0,t_0)} \sim t^{\frac{\sigma}{2-\xi}}$

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$$P_n(\mathbf{X}, t \,|\, \lambda \mathbf{X}_0, 0) \overset{\lambda\uparrow\infty}{\simeq} \sum_{ij} \lambda^{\zeta_{n,p}} \psi_{n,p}(\mathbf{X}, t)\, \phi_{n,p}(\mathbf{X}_0)$$

$$0 = (\partial_t - \mathcal{M}_n) P_n = P_n(\partial_t - \mathcal{M}_n) \quad \text{implies}$$

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$0 = (\partial_t - \mathcal{M}_n) P_n = P_n(\partial_t - \mathcal{M}_n)$ implies

- $\mathcal{M}_n \, \phi_{n,p+1}(\mathbf{X}) = \phi_{n,p}(\mathbf{X})$
- $\partial_t \, \psi_{n,p+1}(\mathbf{X}, t) = -\psi_{n,p}(\mathbf{X}, t)$ (towers of slow modes)
- $\zeta_{n,p} = \zeta_{n,0} + p\,(2 - \xi)$

Particles and Shapes

- Lagrangian average of scaling functions

$$\phi(\lambda\,\mathbf{r}) = \lambda^{\sigma}\phi(\mathbf{r})$$
$$\prec \phi(\boldsymbol{\rho}(t)) \succ = \int d^{\mathfrak{d}_n}\rho\,\phi(\boldsymbol{\rho})\,p_n(\boldsymbol{\rho}, t\,|\,\mathbf{r}, 0)$$

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$$\prec (\mathbf{r}_1 - \mathbf{r}_2)^2 \succ \overset{t\uparrow\infty}{\sim} t^3 \qquad \text{Richardson law}$$

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would imply

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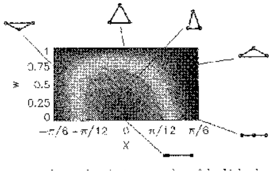
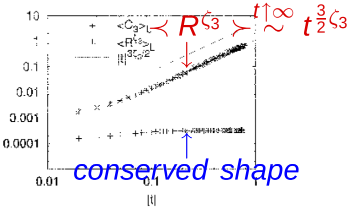
$2\,d$ Navier–Stokes and shapes

Harmonic decomposition of C_3

$$C_3 = \prec R_t^{\zeta} f\,(\chi_t\,, w_t)\,\cos\psi_t \succ + \ldots$$

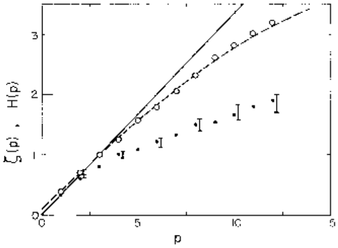




A. Celani & M. Vergassola (2001)

Conclusions

- Genuine anomalous scaling in passive advection from first principles.



Conclusions

- Genuine anomalous scaling in passive advection from first principles.
- Existence of statistical conservation laws observed in realistic models Mydlarski, Pumir, Shraiman, Siggia, and Warhaft (1998), Celani and Vergassola (2001), Celani and Seminara (2006)