A non-perturbative renormalization group study of the stochastic Navier–Stokes equation.

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and
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Outline of the talk

Stochastic Navier–Stokes (SNS) equation with power-law forcing

- Theory.
  1. Kármán-Howarth-Monin equation.
  2. Perturbative renormalization group theory.
- Numerics.
- Open problems.

Non perturbative renormalization group

- What does it mean?
- Application to statistical hydrodynamics.
- Our results.

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SNS: the model

Forster Nelson & Stephen, PRL 36, 867-870 (1976); PRA 16, 732-749 (1977)
De Dominicis & Martin, PRA, 19, 419-422, (1979).

$$(\partial_t + \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} - \kappa \partial^2_{\boldsymbol{x}}) \boldsymbol{v} = \boldsymbol{f} - \partial_{\boldsymbol{x}} P$$

$$\prec\ \boldsymbol{f}(\boldsymbol{x}_1, t_1) \otimes \boldsymbol{f}(\boldsymbol{x}_2, t_2)\ \succ = \delta(t_{12})\, \mathrm{F}(\boldsymbol{x}_{12})$$

$$\mathrm{F}(\boldsymbol{x}_{12}; m, M) = \int_{\mathbb{R}^d} \frac{d^d p}{(2\,\pi)^d} \frac{e^{\imath\, \boldsymbol{p}\cdot\boldsymbol{x}_{12}}\ \mathrm{T}(\boldsymbol{p})}{d-1}\, \check{F}(\boldsymbol{p}; m, M)$$

$$\check{F}(\lambda\, \boldsymbol{p}; \lambda\, m\, , \lambda\, M) = \lambda^{4-d-2\varepsilon}\, \check{F}(\boldsymbol{p}; m, M)$$

$$\mathrm{T}(\boldsymbol{p}) = \mathrm{I} - \boldsymbol{p} \otimes \boldsymbol{p}/p^2$$

Dimensional Analysis

$$\eta_t := [time] \qquad \& \qquad \eta_x := [space]$$

Galilean invariance: match convection with forcing

$$\eta_v - \eta_t = -\frac{\eta_t}{2} - \eta_x (2 - \varepsilon)$$

$$2\,\eta_v - \eta_x = -\frac{\eta_t}{2} - \eta_x (2 - \varepsilon)$$

Solving for η_v and η_t gives

$$\eta_t = 2\,\eta_x \left\{1 - \frac{\varepsilon}{3}\right\}$$

$$\eta_v = \left\{-1 + \frac{2\,\varepsilon}{3}\right\} \eta_x$$

whence the kinetic energy spectrum scaling prediction

$$\eta_\varepsilon = \left\{-1 + \frac{4\,\varepsilon}{3}\right\} \eta_x$$

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Kármán-Howarth-Monin equation for SNS

Homogeneous & isotropic forcing:

$$\boldsymbol{v}_i \equiv \boldsymbol{v}(\boldsymbol{x}_i, t) \qquad i = 1, 2$$

$$\frac{1}{2}\frac{d}{d\,t} \prec \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \succ + \kappa \prec \| \, (\partial_{\boldsymbol{x}} \otimes \boldsymbol{v}) \, \|_{12}^2 \succ - \frac{1}{2}\mathrm{trF} = \frac{1}{4}\partial_{\boldsymbol{x}_1} \cdot \prec (\boldsymbol{v}_1 - \boldsymbol{v}_2) \, \| \, \boldsymbol{v}_1 - \boldsymbol{v}_2 \, \|^2 \succ$$

- it admits an asymptotic inertial range solution under certain hypotheses e.g. Bernard, PRE, 60, 6184-6187 (1999).
- The case $d = 2$ must be a priori distinguished from $d > 2$.

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$d = 3$ ($d > 2$) case

Theoretical predictions from the Kármán-Howarth-Monin equation:

- $\varepsilon < 2$ dimensional scaling dominance.
- $\varepsilon \geq 2$ Kolmogorov constant flux scaling.

Sain, Manu & Pandit, PRL, 81, 4377-4380 (1998)
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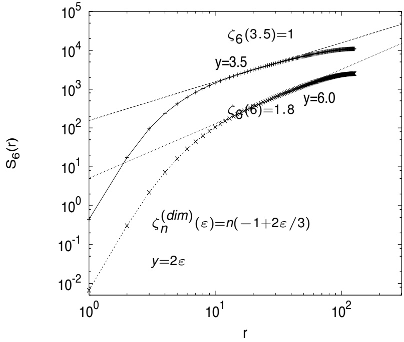

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$d = 2$ case: general theory

Hypotheses encoding Kraichnan's theory (Bernard 1999)

i. velocity correlations are smooth at finite viscosity and exist in the inviscid limit even at coinciding points:

$$\lim_{x \to 0} \prec \boldsymbol{v}(\boldsymbol{x}, t) \cdot v(\boldsymbol{0}, t) \succ = \prec \| \boldsymbol{v} \|^2 (\boldsymbol{0}, t) \succ < \infty$$

ii. Galilean invariant functions and in particular structure functions reach a steady state:

$$\lim_{t \uparrow \infty} \prec (\boldsymbol{v}_1 - \boldsymbol{v}_2) \| \boldsymbol{v}_1 - \boldsymbol{v}_2 \|^2 \succ = \boldsymbol{S}_3(\boldsymbol{x}_{12})$$

iii. Absence of kinetic energy dissipative anomaly:

$$\left\{ \lim_{\nu \downarrow 0} \lim_{x \downarrow 0} - \lim_{x \downarrow 0} \lim_{\nu \downarrow 0} \right\} \kappa \prec \| (\partial_{\boldsymbol{x}} \otimes \boldsymbol{v}) \|_{12}^2 \succ = 0$$

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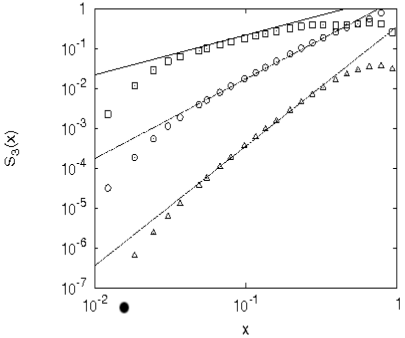
- $0 \leq \varepsilon \leq 2$: energy and enstrophy input U.V. dominated. Inverse cascade.
- $2 < \varepsilon < 3$: I.R. energy input, U.V. enstrophy input. Dimensional scaling.
- $\varepsilon \geq 3$: energy and enstrophy input I.R. dominated. Direct cascade.



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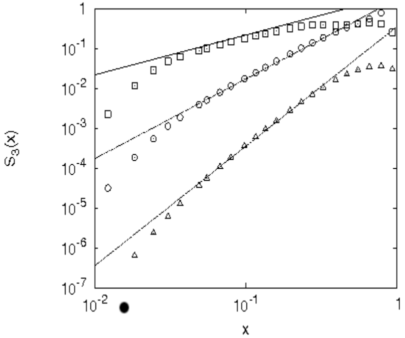

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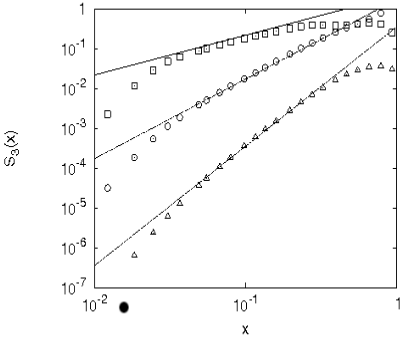

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Expansion in the Hölder exponent ε

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- Gaussian limit by fine-tuning to an $O(\varepsilon)$ forcing correlation amplitude.
- Construction of "small solutions" of the SNS equation.

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- One relevant coupling (eddy diffusivity) for $d > 2$
- Two relevant couplings in $d = 2$ (eddy diffusivity + force amplitude)
- Multiplicatively renormalizable for $d \geq 2$ only if the forcing has a local component.

$$\hat{\mathrm{F}}(p; m, M) = \frac{g_1\, \kappa^3\, h_1(p; M, m)}{p^{d-4+2\,\varepsilon}} + g_2\, \kappa^3\, p^2\, h_2(p; M, m)$$

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$$\mathcal{E}(p) = \varepsilon^{1/3} g_1^{2/3} \kappa^2 p^{1-\frac{4\varepsilon}{3}} R\left(\varepsilon, \frac{m}{p}, \left(\frac{p_b}{p}\right)^{2-\frac{2\varepsilon}{3}}\right)$$

- Agreement for $d = 3$ for $0 \leq \varepsilon \leq 2$.
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operator product expansion scenario

The function R_2 (and analogues for higher order structure functions) do not admit a finite limit as the inverse integral scales $m\,, p_b \downarrow 0$. This is what happens in the homogeneous isotropic Kraichnan model for structure functions of order higher than two.

"Exotic" fixed point scenario

The resummation is not correct. For some critical ε (e.g. $\varepsilon = 0$ if $d = 2$) there appears a non-perturbative mass scale "invalidating" the marginality hypothesis.

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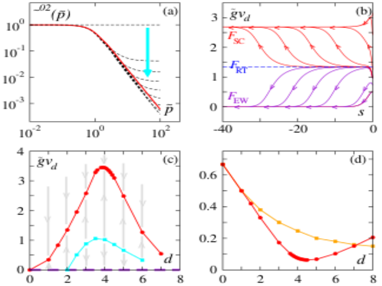
The “exotic” scenario

- Model of interfacial transition: evidence that experimental scaling does not emerge from a bifurcation from the Gaussian fixed point Lipowsky & Fisher PRL, 57, 2411-2414 (1986); PRB, 36, 2126-2141 (1987).
- KPZ equation: evidence of the existence of a non-perturbative fixed point. Canet, Chaté, Delamotte & Wschebor PRL, 104, 150601 (2009)



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601-3

The “exotic” scenario

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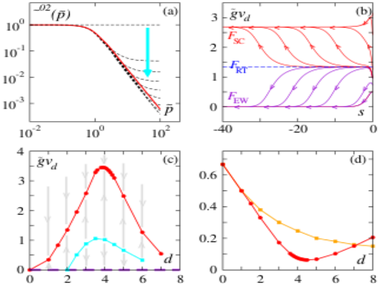

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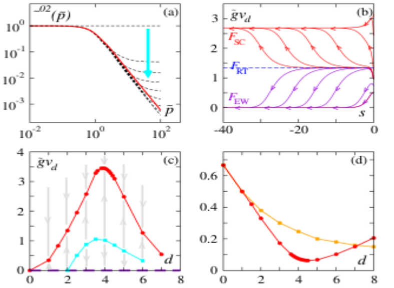

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From the generating function

$$\mathcal{Z}_{(\jmath,\bar{\jmath})} := \prec e^{\jmath \star \boldsymbol{v}(;\boldsymbol{f}+\bar{\jmath})} \succ$$

to the free energy

$$\mathcal{W}_{(\jmath,\bar{\jmath})} := \ln \mathcal{Z}_{(\jmath,\bar{\jmath})}$$

and the thermodynamic potential (average action)

$$\mathcal{U}_{(\boldsymbol{u},\bar{\boldsymbol{u}})} := \sup_{(\jmath,\bar{\jmath})} \left\{ \jmath \star \boldsymbol{u} + \bar{\jmath} \star \bar{\boldsymbol{u}} - \mathcal{W}_{(\jmath,\bar{\jmath})} \right\}$$

“Convex”-envelope of the field theory.

- RG flow: building up of the exact average action $\mathcal{U}$ we decrease an infra-red cut-off.
- Running "high-pass" Gaussian theory

$$\kappa \mapsto \tilde{\kappa} := \kappa + \kappa_{m_r} \check{R}\left(\frac{p}{m_r}\right) \tag{0.4a}$$

$$\boldsymbol{f} \mapsto \tilde{\boldsymbol{f}} \tag{0.4b}$$

with

$$\check{R}(p) = \frac{1}{e^{p^2} - 1} \tag{0.5a}$$

$$\prec \tilde{\boldsymbol{f}}_1 \otimes \tilde{\boldsymbol{f}}_2 \succ = \delta(t_{12}) \left[\frac{F_o\, h_1(p; m_r)}{p^{d-4+2\,\varepsilon}} + F_{m_r}\, p^2\, e^{-\frac{p^2}{m_r}} \right]$$

- κ_{m_r}, F_{m_r} fine-tuning parameters.

How to investigate: non-perturbative RG analysis I

Exact flow equations

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Saddle point solution as initial condition

$$\mathcal{U}_M \sim \bar{\boldsymbol{u}} \star (\partial_t + \boldsymbol{u} \cdot \partial_{\boldsymbol{x}} - \kappa\, \partial_x^2)\boldsymbol{u} - \frac{\bar{\boldsymbol{u}} \star \mathrm{F} \star \bar{\boldsymbol{u}}}{2} \tag{0.6a}$$

$$\partial_{\boldsymbol{x}} \cdot \boldsymbol{u} = \partial_{\boldsymbol{x}} \cdot \bar{\boldsymbol{u}} = 0 \tag{0.6b}$$

and flow equations

$$m_r \partial_{m_r} \left\{ \mathcal{U}_{(\boldsymbol{u},\bar{\boldsymbol{u}})} - \frac{1}{2} [\boldsymbol{u}\,,\bar{\boldsymbol{u}}] \star (m_r \partial_{m_r} \mathcal{R}) \star \begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix} \right\} =$$

$$-\mathrm{tr}\, \frac{(m_r \partial_{m_r} \mathcal{R})}{2} \star \left[\sum_{n=0}^{\infty} (-\mathbf{W}^{(2)} \star \mathcal{U}^{(2)int}_{(\boldsymbol{u},\bar{\boldsymbol{u}})})^n \right] \star \mathbf{W}^{(2)}$$

- The exact flow equation correspond to an infinite hierarchy of equations.
- The physical system is specified by the initial condition.
- The scope is not to reconstruct the original theory.
- The scope is limited to scaling analysis (universality class).
- We can look for solutions of the flow equations using
  1. Galilean symmetry
  2. Perturbative “relevance” (Wilsonian sense)

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How to investigate: non-perturbative RG analysis III

Truncations

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The simplest truncation: I.R. equal time-sector of the theory (analogue of Canet et al. PRL, 104, 150601 (2009))

$$\mathcal{U}_{(\boldsymbol{u},\bar{\boldsymbol{u}})} = \boldsymbol{u} \star \mathrm{U}^{(1,1)} \star \bar{\boldsymbol{u}} + \frac{1}{2}\mathrm{U}^{(0,2)}(\star\bar{\boldsymbol{u}})^2 + \frac{1}{2}(\boldsymbol{u}\star)^2\mathrm{U}^{(2,1)} \star \bar{\boldsymbol{u}}$$

$$\frac{\check{\mathrm{U}}^{(1,1)}(\boldsymbol{p}_1,\omega_1|\boldsymbol{p}_2,\omega_2)}{(2\,\pi)^{d+1}\,\delta^{(d)}(\sum_{i=1}^{2}\boldsymbol{p}_i)\delta(\sum_{i=1}^{2}\omega_i)\,\mathrm{T}(\boldsymbol{p}_1)} = \imath\,\omega_1 + \kappa_{m_r}\,p_1^2\left[\gamma^{(1,1)}\,(p;m_r) + \check{R}\left(\frac{p}{m_r}\right)\right]$$

$$\frac{\check{\mathrm{U}}^{(0,2)}(\boldsymbol{p}_1,\omega_1,\boldsymbol{p}_2,\omega_2)}{(2\,\pi)^{d+1}\,\delta^{(d)}(\sum_{i=1}^{2}\boldsymbol{p}_i)\delta(\sum_{i=1}^{2}\omega_i)\,\mathrm{T}(\boldsymbol{p}_1)} = -\left[\frac{\sum_{i=0}^{1}\mathrm{tr}\,\check{\mathrm{F}}_{(i)}(p_1,m_r)}{d-1} + \left(\lambda_{(0)}\,m_r^{2-d-2\,\varepsilon} + \lambda_{(1)}\right)\,p^2\,\gamma^{(0,2)}\,(p;m_r)\right]$$

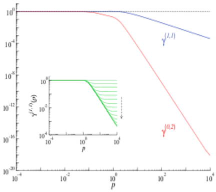
The truncated flow

$$
\begin{aligned}
&[m_r\partial_{m_r} - \boldsymbol{p}\cdot\partial_{\boldsymbol{p}} + \eta_\kappa]\gamma^{(1,1)}(p) \\
&\qquad = \eta_F\, G_F^{(1,1)}(p) - \eta_\kappa\, G_\kappa^{(1,1)}(p) - G_o^{(1,1)}(p) \\
&[m_r\partial_{m_r} - \boldsymbol{p}\cdot\partial_{\boldsymbol{p}} + \tilde{\eta}_F]\,\gamma^{(0,2)}(p) \\
&\qquad = \eta_F\, G_F^{(0,2)}(p) - \eta_\kappa\, G_\kappa^{(0,2)}(p) - G_o^{(0,2)}(p) \\
&m_r\partial_{m_r}\lambda_{(0)} = -\lambda_{(0)}\,(3\,\eta_k + 2\,\varepsilon) \\
&m_r\partial_{m_r}\lambda_{(1)} = -\lambda_{(1)}\,(3\,\eta_k + 2 - d - \eta_F) \\
&\gamma^{(1,1)}(p_o) = 1 \qquad\qquad \text{(eddy diffusivity renorm. cond.)} \\
&\gamma^{(0,2)}(p_o) = 1 \qquad\qquad \text{(local forcing amplitude renorm. cond.)}
\end{aligned}
$$

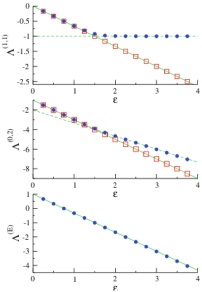
and

$$
\tilde{\eta}_F = \frac{(2 - d - 2\,\varepsilon)\lambda_{(0)} + \eta_F\,\lambda_{(1)}}{\left(\lambda_{(0)} + \lambda_{(1)}\right)}
$$

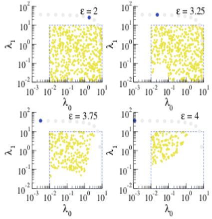
Results



$\gamma^{(1,1)}$
$\gamma^{(0,2)}$
p

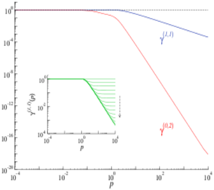


$\Lambda^{(1,1)}$
$\Lambda^{(0,2)}$
$\Lambda^{(E)}$
ε



$\varepsilon = 2$
$\varepsilon = 3.25$
$\varepsilon = 3.75$
$\varepsilon = 4$
λ_1
λ_0

Results



$\gamma^{(1,1)}$
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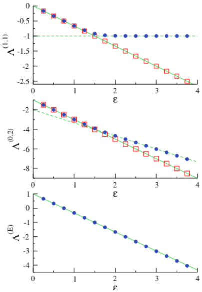

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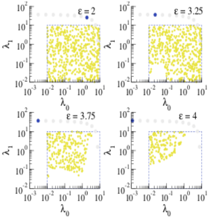

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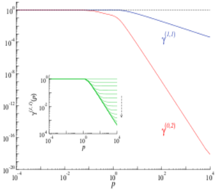

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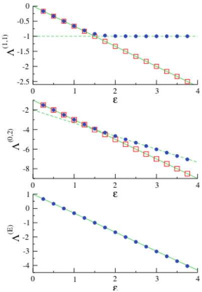

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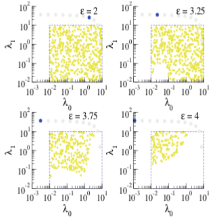

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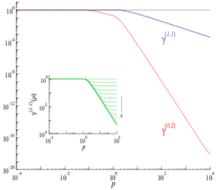

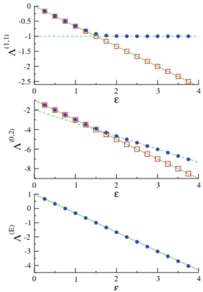

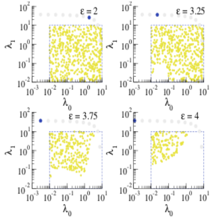

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