Optimal control of non-equilibrium processes in stochastic thermodynamics.

Paolo Muratore-Ginanneschi [1]

[1] Department of Mathematics and Statistics
University of Helsinki.
https://wiki.helsinki.fi/display/mathphys/paolo

Aalto University, Helsinki, August 2012

Outline of the talk

Background: from time reversal of Markov Processes to Stochastic Thermodynamics

- Schrödinger & Nelson. Scope: interpretation of Q.M.
- Kolmogorov: from time reversal to detailed balance.
- A recent different application: non-equilibrium thermodynamics of small systems.
- The "Cost" of deterministic time reversal

Control of thermodynamic processes

- Over-damped dynamics: Work, Heat, Entropy production.
- "Coercivity" and control.
- A simple "universal" result.

Disclaimer:

informal discussion without pretense of mathematical rigor!

Outline of the talk

Background: from time reversal of Markov Processes to Stochastic Thermodynamics

- Schrödinger & Nelson. Scope: interpretation of Q.M.
- Kolmogorov: from time reversal to detailed balance.
- A recent different application: non-equilibrium thermodynamics of small systems.
- The “Cost” of deterministic time reversal

Control of thermodynamic processes

- Over-damped dynamics: Work, Heat, Entropy production.
- “Coercivity” and control.
- A simple “universal” result.

Disclaimer:

informal discussion without pretense of mathematical rigor!

Outline of the talk

Background: from time reversal of Markov Processes to Stochastic Thermodynamics

- Schrödinger & Nelson. Scope: interpretation of Q.M.
- Kolmogorov: from time reversal to detailed balance.
- A recent different application: non-equilibrium thermodynamics of small systems.
- The "Cost" of deterministic time reversal

Control of thermodynamic processes

- Over-damped dynamics: Work, Heat, Entropy production.
- "Coercivity" and control.
- A simple "universal" result.

Disclaimer:

informal discussion without pretense of mathematical rigor!

Outline of the talk

Background: from time reversal of Markov Processes to Stochastic Thermodynamics

- Schrödinger & Nelson. Scope: interpretation of Q.M.
- Kolmogorov: from time reversal to detailed balance.
- A recent different application: non-equilibrium thermodynamics of small systems.
- The "Cost" of deterministic time reversal

Control of thermodynamic processes

- Over-damped dynamics: Work, Heat, Entropy production.
- "Coercivity" and control.
- A simple "universal" result.

Disclaimer:

informal discussion without pretense of mathematical rigor!

Outline of the talk

Background: from time reversal of Markov Processes to Stochastic Thermodynamics

- Schrödinger & Nelson. Scope: interpretation of Q.M.
- Kolmogorov: from time reversal to detailed balance.
- A recent different application: non-equilibrium thermodynamics of small systems.
- The “Cost” of deterministic time reversal

Control of thermodynamic processes

- Over-damped dynamics: Work, Heat, Entropy production.
- “Coercivity” and control.
- A simple “universal” result.

Disclaimer:

informal discussion without pretense of mathematical rigor!

Outline of the talk

Background: from time reversal of Markov Processes to Stochastic Thermodynamics

- Schrödinger & Nelson. Scope: interpretation of Q.M.
- Kolmogorov: from time reversal to detailed balance.
- A recent different application: non-equilibrium thermodynamics of small systems.
- The “Cost” of deterministic time reversal

Control of thermodynamic processes

- Over-damped dynamics: Work, Heat, Entropy production.
- “Coercivity” and control.
- A simple “universal” result.

Disclaimer:

informal discussion without pretense of mathematical rigor!

Outline of the talk

Background: from time reversal of Markov Processes to Stochastic Thermodynamics

- Schrödinger & Nelson. Scope: interpretation of Q.M.
- Kolmogorov: from time reversal to detailed balance.
- A recent different application: non-equilibrium thermodynamics of small systems.
- The "Cost" of deterministic time reversal

Control of thermodynamic processes

- Over-damped dynamics: Work, Heat, Entropy production.
- "Coercivity" and control.
- A simple "universal" result.

Disclaimer:

informal discussion without pretense of mathematical rigor!

Outline of the talk

Background: from time reversal of Markov Processes to Stochastic Thermodynamics

- Schrödinger & Nelson. Scope: interpretation of Q.M.
- Kolmogorov: from time reversal to detailed balance.
- A recent different application: non-equilibrium thermodynamics of small systems.
- The "Cost" of deterministic time reversal

Control of thermodynamic processes

- Over-damped dynamics: Work, Heat, Entropy production.
- "Coercivity" and control.
- A simple "universal" result.

Disclaimer:

informal discussion without pretense of mathematical rigor!

Outline of the talk

Background: from time reversal of Markov Processes to Stochastic Thermodynamics

- Schrödinger & Nelson. Scope: interpretation of Q.M.
- Kolmogorov: from time reversal to detailed balance.
- A recent different application: non-equilibrium thermodynamics of small systems.
- The "Cost" of deterministic time reversal

Control of thermodynamic processes

- Over-damped dynamics: Work, Heat, Entropy production.
- "Coercivity" and control.
- A simple "universal" result.

Disclaimer:

informal discussion without pretense of mathematical rigor!

Outline of the talk

Background: from time reversal of Markov Processes to Stochastic Thermodynamics

- Schrödinger & Nelson. Scope: interpretation of Q.M.
- Kolmogorov: from time reversal to detailed balance.
- A recent different application: non-equilibrium thermodynamics of small systems.
- The "Cost" of deterministic time reversal

Control of thermodynamic processes

- Over-damped dynamics: Work, Heat, Entropy production.
- "Coercivity" and control.
- A simple "universal" result.

Disclaimer:

informal discussion without pretense of mathematical rigor!

Schrödinger's diffusion problem

E. Schrödinger. "Über die Umkehrung der Naturgesetze". In: *Sitzungsberichte der preussischen Akademie der Wissenschaften, physikalische mathematische Klasse* 8.9 (1931), pp. 144–153. DOI: 10.1002/ange.19310443014

- Given two probability densities

$$\mathrm{m}_o(d^d x) = \mathrm{m}(d^d x, t_o) \qquad \& \qquad \mathrm{m}_f(d^d x) = \mathrm{m}(d^d x, t_f)$$

  at the end-points of a time interval $[t_o, t_f]$,
- find the interpolating diffusion

$$\mathrm{m}(d^d x, t) \qquad \forall\, t \in [t_o, t_f]$$

Schrödinger's diffusion problem

E. Schrödinger. "Über die Umkehrung der Naturgesetze". In: *Sitzungsberichte der preussischen Akademie der Wissenschaften, physikalische mathematische Klasse* 8.9 (1931), pp. 144–153. DOI: 10.1002/ange.19310443014

- Given two probability densities

$$\mathrm{m}_o(d^d x) = \mathrm{m}(d^d x, t_o) \qquad \& \qquad \mathrm{m}_f(d^d x) = \mathrm{m}(d^d x, t_f)$$

  at the end-points of a time interval $[t_o, t_f]$,
- find the interpolating diffusion

$$\mathrm{m}(d^d x, t) \qquad \forall\ t \in [t_o, t_f]$$

Reciprocal diffusion processes

Let $\{\xi_t; t \in [t_o, t_f]\}$ be an e.g. $\mathbb{R}^d$-valued diffusion process

$$\mathrm{p}(d^d x, t|\boldsymbol{x}_1, t_1) = \mathrm{P}(\xi_t \in [\boldsymbol{x}, \boldsymbol{x} + d\boldsymbol{x}]|\xi_{t_1} = \boldsymbol{x}_1)$$

Schrödinger's (math-refinements:[1] see also e.g.[2]) finding:

$$\mathrm{q}(d^d x_n, t_n, \dots, d^d x_1, t_1|\mathrm{m}_o, \mathrm{m}_f) = \psi[m_o, m_f](d^d x_o)\, \bar{\psi}[m_o, m_f](d^d x_f) \prod_{i=1}^{n-1} \mathrm{p}(d^d x_{i+1}, t|\boldsymbol{x}_i, t_i)$$

The auxiliary scalar fields

$$\psi[m_o, m_f](d^d x_o) \qquad \& \qquad \bar{\psi}[m_o, m_f](d^d x_f)$$

are Lagrange multipliers imposing the boundary conditions.

---

[1] Bernstein. 1932; Jamison. 1974.

[2] Dai Pra. 1991; Krener. 1997.

Kolmogorov analysis of time-reversal

A. N. Kolmogorov. "Zur Umkehrbarkeit der statistischen Naturgesetze". In: *Mathematische Annalen* 113 (1 1937), pp. 766–772. DOI: 10.1007/BF01571664

- Fix two densities $\mathrm{m}_o, \mathrm{m}_f$ at the end-points of $[t_i, t_f]$.
- The time reversed Markov evolution must satisfy[3] for any $t_o \leq t_1 \leq t_2 \leq t_f$:

$$\mathrm{m}(\boldsymbol{x}_2, t_2)\, \tilde{\mathrm{p}}(\boldsymbol{x}_1, t_1 | \boldsymbol{x}_2, t_2) = \mathrm{p}(\boldsymbol{x}_2, t_2 | \boldsymbol{x}_1, t_1)\, \mathrm{m}(\boldsymbol{x}_1, t_1)$$

- Add the hypotheses
  - the diffusion is time-stationary: $\mathrm{p}(\boldsymbol{x}_2, t_2 | \boldsymbol{x}_1, t_1) = \mathrm{p}(\boldsymbol{x}_2, t_2 - t_1 | \boldsymbol{x}_1, 0)$
  - $\mathrm{m}(\boldsymbol{x}, t) = \mathrm{m}_*(\boldsymbol{x}) > 0$ is stationary.

Question: under which conditions

$$\tilde{\mathrm{p}}(\boldsymbol{x}_1, 0 | \boldsymbol{x}_2, t_2 - t_1) = \mathrm{p}(\boldsymbol{x}_1, t_2 - t_1 | \boldsymbol{x}_2, 0)$$

[3] Kolmogorov. 1936.

Kolmogorov analysis of time-reversal

A. N. Kolmogorov. "Zur Umkehrbarkeit der statistischen Naturgesetze". In: *Mathematische Annalen* 113 (1 1937), pp. 766–772. DOI: 10.1007/BF01571664

- Fix two densities $\mathrm{m}_o, \mathrm{m}_f$ at the end-points of $[t_i, t_f]$.
- The time reversed Markov evolution must satisfy[3] for any $t_o \leq t_1 \leq t_2 \leq t_f$:

$$\mathrm{m}(\boldsymbol{x}_2, t_2)\,\tilde{\mathrm{p}}(\boldsymbol{x}_1, t_1|\boldsymbol{x}_2, t_2) = \mathrm{p}(\boldsymbol{x}_2, t_2|\boldsymbol{x}_1, t_1)\,\mathrm{m}(\boldsymbol{x}_1, t_1)$$

- Add the hypotheses
  - the diffusion is time-stationary: $\mathrm{p}(\boldsymbol{x}_2, t_2|\boldsymbol{x}_1, t_1) = \mathrm{p}(\boldsymbol{x}_2, t_2 - t_1|\boldsymbol{x}_1, 0)$
  - $\mathrm{m}(\boldsymbol{x}, t) = \mathrm{m}_*(\boldsymbol{x}) > 0$ is stationary.

Question: under which conditions

$$\tilde{\mathrm{p}}(\boldsymbol{x}_1, 0|\boldsymbol{x}_2, t_2 - t_1) = \mathrm{p}(\boldsymbol{x}_1, t_2 - t_1|\boldsymbol{x}_2, 0)$$

[3] Kolmogorov. 1936.

Kolmogorov analysis of time-reversal

A. N. Kolmogorov. "Zur Umkehrbarkeit der statistischen Naturgesetze". In: *Mathematische Annalen* 113 (1 1937), pp. 766–772. DOI: 10.1007/BF01571664

- Fix two densities $\mathrm{m}_o, \mathrm{m}_f$ at the end-points of $[t_i, t_f]$.
- The time reversed Markov evolution must satisfy[3] for any $t_o \leq t_1 \leq t_2 \leq t_f$:

$$\mathrm{m}(\boldsymbol{x}_2, t_2)\, \tilde{\mathrm{p}}(\boldsymbol{x}_1, t_1 | \boldsymbol{x}_2, t_2) = \mathrm{p}(\boldsymbol{x}_2, t_2 | \boldsymbol{x}_1, t_1)\, \mathrm{m}(\boldsymbol{x}_1, t_1)$$

- Add the hypotheses
  - the diffusion is time-stationary: $\mathrm{p}(\boldsymbol{x}_2, t_2 | \boldsymbol{x}_1, t_1) = \mathrm{p}(\boldsymbol{x}_2, t_2 - t_1 | \boldsymbol{x}_1, 0)$
  - $\mathrm{m}(\boldsymbol{x}, t) = \mathrm{m}_\star(\boldsymbol{x}) > 0$ is stationary.

Question: under which conditions

$$\tilde{\mathrm{p}}(\boldsymbol{x}_1, 0 | \boldsymbol{x}_2, t_2 - t_1) = \mathrm{p}(\boldsymbol{x}_1, t_2 - t_1 | \boldsymbol{x}_2, 0)$$

[3] Kolmogorov. 1936.

Kolmogorov analysis of time-reversal

A. N. Kolmogorov. "Zur Umkehrbarkeit der statistischen Naturgesetze". In: *Mathematische Annalen* 113 (1 1937), pp. 766–772. DOI: 10.1007/BF01571664

- Fix two densities $\mathrm{m}_o, \mathrm{m}_f$ at the end-points of $[t_i, t_f]$.
- The time reversed Markov evolution must satisfy[3] for any $t_o \leq t_1 \leq t_2 \leq t_f$:

$$\mathrm{m}(\boldsymbol{x}_2, t_2)\,\tilde{\mathrm{p}}(\boldsymbol{x}_1, t_1|\boldsymbol{x}_2, t_2) = \mathrm{p}(\boldsymbol{x}_2, t_2|\boldsymbol{x}_1, t_1)\,\mathrm{m}(\boldsymbol{x}_1, t_1)$$

- Add the hypotheses
  - the diffusion is time-stationary: $\mathrm{p}(\boldsymbol{x}_2, t_2|\boldsymbol{x}_1, t_1) = \mathrm{p}(\boldsymbol{x}_2, t_2 - t_1|\boldsymbol{x}_1, 0)$
  - $\mathrm{m}(\boldsymbol{x}, t) = \mathrm{m}_\star(\boldsymbol{x}) > 0$ is stationary.

Question: under which conditions

$$\tilde{\mathrm{p}}(\boldsymbol{x}_1, 0|\boldsymbol{x}_2, t_2 - t_1) = \mathrm{p}(\boldsymbol{x}_1, t_2 - t_1|\boldsymbol{x}_2, 0)$$

[3] Kolmogorov. 1936.

Kolmogorov's covariant characterization of a diffusion

Multiplicative noise warning:

Even for an $\mathbb{R}^d$-valued process ξ the scale of the noise imposes a "Riemannian" metric $\mathrm{g} : d\boldsymbol{x} \otimes d\boldsymbol{x}$ on the space.

$$\lim_{dt\downarrow 0} \mathrm{E}\left\{ \frac{\xi_{t+dt} - \xi_t}{dt} \middle| \xi_t = \boldsymbol{x} \right\} = \left(\boldsymbol{f} - \frac{1}{2}\Gamma : \mathrm{g}^{-1} \right)(\boldsymbol{x})$$

$$\lim_{dt\downarrow 0} \mathrm{E}\left\{ \frac{(\xi_{t+dt} - \xi_t) \otimes (\xi_{t+dt} - \xi_t)}{dt} \middle| \xi_t = \boldsymbol{x} \right\} = \mathrm{g}^{-1}(\boldsymbol{x})$$

Kolmogorov's result

$$\tilde{\mathrm{p}}(\boldsymbol{x}_1, 0|\boldsymbol{x}_2, t_2 - t_1) = \mathrm{p}(\boldsymbol{x}_1, t_2 - t_1|\boldsymbol{x}_2, 0) \quad \Leftrightarrow \quad \boldsymbol{f}(\boldsymbol{x}) = -\mathrm{g}^{-1} \cdot (\partial_{\boldsymbol{x}} U)(\boldsymbol{x})$$

Kolmogorov's covariant characterization of a diffusion

Multiplicative noise warning:

Even for an $\mathbb{R}^d$-valued process ξ the scale of the noise imposes a "Riemannian" metric $\mathrm{g} : d\boldsymbol{x} \otimes d\boldsymbol{x}$ on the space.

$$\lim_{dt \downarrow 0} \mathrm{E}\left\{ \frac{\xi_{t+dt} - \xi_t}{dt} \middle| \xi_t = \boldsymbol{x} \right\} = \left(\boldsymbol{f} - \frac{1}{2} \Gamma : \mathrm{g}^{-1} \right)(\boldsymbol{x})$$

$$\lim_{dt \downarrow 0} \mathrm{E}\left\{ \frac{(\xi_{t+dt} - \xi_t) \otimes (\xi_{t+dt} - \xi_t)}{dt} \middle| \xi_t = \boldsymbol{x} \right\} = \mathrm{g}^{-1}(\boldsymbol{x})$$

Kolmogorov's result

$$\tilde{\mathrm{p}}(\boldsymbol{x}_1, 0 | \boldsymbol{x}_2, t_2 - t_1) = \mathrm{p}(\boldsymbol{x}_1, t_2 - t_1 | \boldsymbol{x}_2, 0) \quad \Leftrightarrow \quad \boldsymbol{f}(\boldsymbol{x}) = -\mathrm{g}^{-1} \cdot (\partial_{\boldsymbol{x}} U)(\boldsymbol{x})$$

Kolmogorov's covariant characterization of a diffusion

Multiplicative noise warning:

Even for an $\mathbb{R}^d$-valued process ξ the scale of the noise imposes a "Riemannian" metric $\mathrm{g} : d\boldsymbol{x} \otimes d\boldsymbol{x}$ on the space.

Christoffel symbols of g

$$\lim_{dt\downarrow 0} \mathrm{E}\left\{ \frac{\xi_{t+dt} - \xi_t}{dt} \middle| \xi_t = \boldsymbol{x} \right\} = \left(\boldsymbol{f} - \frac{1}{2} \Gamma : \mathrm{g}^{-1} \right)(\boldsymbol{x})$$

here: strictly positive definite

$$\lim_{dt\downarrow 0} \mathrm{E}\left\{ \frac{(\xi_{t+dt} - \xi_t) \otimes (\xi_{t+dt} - \xi_t)}{dt} \middle| \xi_t = \boldsymbol{x} \right\} = \mathrm{g}^{-1}(\boldsymbol{x})$$

Kolmogorov's result

$$\tilde{\mathrm{p}}(\boldsymbol{x}_1, 0 | \boldsymbol{x}_2, t_2 - t_1) = \mathrm{p}(\boldsymbol{x}_1, t_2 - t_1 | \boldsymbol{x}_2, 0) \quad \Leftrightarrow \quad \boldsymbol{f}(\boldsymbol{x}) = -\mathrm{g}^{-1} \cdot (\partial_{\boldsymbol{x}} U)(\boldsymbol{x})$$

Kolmogorov's covariant characterization of a diffusion

Multiplicative noise warning:

Even for an $\mathbb{R}^d$-valued process ξ the scale of the noise imposes a "Riemannian" metric $g : d\boldsymbol{x} \otimes d\boldsymbol{x}$ on the space.

Christoffel symbols of g

$$\lim_{dt \downarrow 0} \mathrm{E}\left\{ \frac{\xi_{t+dt} - \xi_t}{dt} \middle| \xi_t = \boldsymbol{x} \right\} = \left(\boldsymbol{f} - \frac{1}{2} \Gamma : g^{-1} \right)(\boldsymbol{x})$$

here: strictly positive definite

$$\lim_{dt \downarrow 0} \mathrm{E}\left\{ \frac{(\xi_{t+dt} - \xi_t) \otimes (\xi_{t+dt} - \xi_t)}{dt} \middle| \xi_t = \boldsymbol{x} \right\} = g^{-1}(\boldsymbol{x})$$

Kolmogorov's result

$$\tilde{\mathrm{p}}(\boldsymbol{x}_1, 0 | \boldsymbol{x}_2, t_2 - t_1) = \mathrm{p}(\boldsymbol{x}_1, t_2 - t_1 | \boldsymbol{x}_2, 0) \quad \Leftrightarrow \quad \boldsymbol{f}(\boldsymbol{x}) = -g^{-1} \cdot (\partial_{\boldsymbol{x}} U)(\boldsymbol{x})$$

Edward Nelson's time reversal for smooth diffusions

E. Nelson. *Dynamical Theories of Brownian Motion.* second edition. Princeton University Press, 2001, p. 148

Mean backward derivatives

$$\lim_{dt\downarrow 0} \mathrm{E}\left\{ \frac{\xi_t - \xi_{t-dt}}{dt} \middle| \xi_t = \boldsymbol{x} \right\} = \left(\tilde{\boldsymbol{f}} + \frac{1}{2}\Gamma : \mathrm{g}^{-1} \right)(\boldsymbol{x})$$

$$\lim_{dt\downarrow 0} \mathrm{E}\left\{ \frac{(\xi_t - \xi_{t-dt}) \otimes (\xi_t - \xi_{t-dt})}{dt} \middle| \xi_t = \boldsymbol{x} \right\} = \mathrm{g}^{-1}(\boldsymbol{x})$$

- Assigning $(\mathrm{m}_o, \mathrm{m}_f)$ at $t_o \leq t_f$ specifies $\tilde{\boldsymbol{f}}$ in terms of $\boldsymbol{f}$.
- Ito lemma in local coordinates becomes ($d\xi_t = \xi_t - \xi_{t-dt}$)

$$F(\xi_t, t) - F(\xi_{t-dt}, t - dt) = d\xi_t \cdot \partial_{\xi_t} F(\xi_t, t)$$
$$+ dt\, \partial_t F(\xi_t, t) - \frac{1}{2} d \prec \xi_t \otimes \xi_t \succ : \partial_{\xi_t} \otimes \partial_{\xi_t} F(\xi_t, t)$$

Edward Nelson's time reversal for smooth diffusions

E. Nelson. *Dynamical Theories of Brownian Motion*. second edition. Princeton University Press, 2001, p. 148

Mean backward derivatives

$$\lim_{dt \downarrow 0} \mathrm{E}\left\{ \frac{\xi_t - \xi_{t-dt}}{dt} \middle| \xi_t = \boldsymbol{x} \right\} = \left(\tilde{\boldsymbol{f}} + \frac{1}{2}\Gamma : \mathrm{g}^{-1} \right)(\boldsymbol{x})$$

$$\lim_{dt \downarrow 0} \mathrm{E}\left\{ \frac{(\xi_t - \xi_{t-dt}) \otimes (\xi_t - \xi_{t-dt})}{dt} \middle| \xi_t = \boldsymbol{x} \right\} = \mathrm{g}^{-1}(\boldsymbol{x})$$

- Assigning $(\mathrm{m}_o, \mathrm{m}_f)$ at $t_o \leq t_f$ specifies $\tilde{\boldsymbol{f}}$ in terms of $\boldsymbol{f}$.
- Ito lemma in local coordinates becomes $(d\xi_t = \xi_t - \xi_{t-dt})$

$$F(\xi_t, t) - F(\xi_{t-dt}, t - dt) = d\xi_t \cdot \partial_{\xi_t} F(\xi_t, t)$$
$$+ dt\, \partial_t F(\xi_t, t) - \frac{1}{2} d \prec \xi_t \otimes \xi_t \succ : \partial_{\xi_t} \otimes \partial_{\xi_t} F(\xi_t, t)$$

Edward Nelson's time reversal for smooth diffusions

E. Nelson. *Dynamical Theories of Brownian Motion*. second edition. Princeton University Press, 2001, p. 148

Mean backward derivatives

Christoffel symbols of g

$$\lim_{dt\downarrow 0} \mathrm{E}\left\{ \frac{\xi_t - \xi_{t-dt}}{dt} \middle| \xi_t = \boldsymbol{x} \right\} = \left(\tilde{\boldsymbol{f}} + \frac{1}{2}\Gamma : \mathrm{g}^{-1} \right)(\boldsymbol{x})$$

$$\lim_{dt\downarrow 0} \mathrm{E}\left\{ \frac{(\xi_t - \xi_{t-dt}) \otimes (\xi_t - \xi_{t-dt})}{dt} \middle| \xi_t = \boldsymbol{x} \right\} = \mathrm{g}^{-1}(\boldsymbol{x})$$

- Assigning $(\mathrm{m}_o, \mathrm{m}_f)$ at $t_o \le t_f$ specifies $\tilde{\boldsymbol{f}}$ in terms of $\boldsymbol{f}$.
- Ito lemma in local coordinates becomes ($d\xi_t = \xi_t - \xi_{t-dt}$)

$$F(\xi_t, t) - F(\xi_{t-dt}, t - dt) = d\xi_t \cdot \partial_{\xi_t} F(\xi_t, t)$$
$$+ dt\, \partial_t F(\xi_t, t) - \frac{1}{2} d \prec \xi_t \otimes \xi_t \succ : \partial_{\xi_t} \otimes \partial_{\xi_t} F(\xi_t, t)$$

Backward drift via Girsanov formula: covariant Wiener measure

Suppose as above that g be

- strictly positive definite;
- time independent.

The (Eells-Elworthy[4]) development map

$$d\omega_t = \mathrm{e}_t \overset{\diamond}{\cdot} d\beta_t$$

$$d\mathrm{e}_t = -\Gamma : \mathrm{e}_t \overset{\diamond}{\otimes} d\omega_t$$

gives a covariant description of a *d*-dimensional Riemann manifold-valued Wiener process.

4 Eells and Elworthy. 1970.

Backward drift via Girsanov formula: covariant Wiener measure

Suppose as above that g be

- strictly positive definite;
- time independent.

The (Eells-Elworthy[4]) development map

Stratonovich differential

$$d\omega_t = e_t \cdot \diamond\, d\beta_t$$

Standard "Euclidean" Brownian motion

Christoffel symbols of g

Stratonovich differential

$$de_t = -\Gamma : e_t \overset{\diamond}{\otimes} d\omega_t$$

gives a covariant description of a d-dimensional Riemann manifold-valued Wiener process.

4 Eells and Elworthy. 1970.

Backward drift via Girsanov formula: covariant Wiener measure

Suppose as above that g be

- strictly positive definite;
- time independent.

The (Eells-Elworthy[4]) development map

Stratonovich differential

$$d\omega_t = e_t \overset{\diamond}{\cdot} d\beta_t$$

Standard "Euclidean" Brownian motion

Christoffel symbols of g

Stratonovich differential

$$de_t = -\Gamma : e_t \overset{\diamond}{\otimes} d\omega_t$$

gives a covariant description of a d-dimensional Riemann manifold-valued Wiener process.

The important aspect here is covariance, the manifold may well be $\mathbb{R}^d$

4 Eells and Elworthy. 1970.

Backward drift via Girsanov formula

- The Wiener process $\omega \equiv \{\omega_t; t \in [t_o, t_f]\}$ is invariant under time reversal.
- Semi-martingale $\xi \equiv \{\xi_t; t \in [t_o, t_f]\}$ absolutely continuous w.r.t. ω with non-vanishing covariant drift $\boldsymbol{f}$. Girsanov formula permits to write for any n-tuple $t_o \leq t_1 \leq \dots t_n \leq t_f$

$$\mathrm{E}F(\xi_{t_1}, \dots, \xi_{t_n}) = \mathrm{E}\frac{d\mathrm{P}_\xi}{d\mathrm{P}_\omega} F(\omega_{t_1}, \dots, \omega_{t_n})$$

- Let $(\mathrm{m}_o = \sqrt{\det \mathrm{g}}\, \mathrm{n}_o, \; \mathrm{m}_f := \sqrt{\det \mathrm{g}}\, \mathrm{n}_f)$ at $t_o \leq t_f$ be assigned (as before)

scalar density: w.r.t. the invariant volume

Backward drift via Girsanov formula

- The Wiener process $\omega \equiv \{\omega_t; t \in [t_o, t_f]\}$ is invariant under time reversal.
- Semi-martingale $\xi \equiv \{\xi_t; t \in [t_o, t_f]\}$ absolutely continuous w.r.t. ω with non-vanishing covariant drift $\boldsymbol{f}$. Girsanov formula permits to write for any n-tuple $t_o \leq t_1 \leq \ldots t_n \leq t_f$

$$\mathrm{E} F(\xi_{t_1}, \ldots, \xi_{t_n}) = \mathrm{E} \frac{d\mathrm{P}_\xi}{d\mathrm{P}_\omega} F(\omega_{t_1}, \ldots, \omega_{t_n})$$

- Let $(\mathrm{m}_o = \sqrt{\det \mathrm{g}}\,\mathrm{n}_o\,,\ \mathrm{m}_f := \sqrt{\det \mathrm{g}}\,\mathrm{n}_f)$ at $t_o \leq t_f$ be assigned (as before)

scalar density: w.r.t. the invariant volume

Backward drift via Girsanov formula

- The Wiener process $\omega \equiv \{\omega_t; t \in [t_o, t_f]\}$ is invariant under time reversal.
- Semi-martingale $\xi \equiv \{\xi_t; t \in [t_o, t_f]\}$ absolutely continuous w.r.t. ω with non-vanishing covariant drift $\boldsymbol{f}$. Girsanov formula permits to write for any n-tuple $t_o \leq t_1 \leq \ldots t_n \leq t_f$

$$\mathrm{E} F(\xi_{t_1}, \ldots, \xi_{t_n}) = \mathrm{E} \frac{d\mathrm{P}_\xi}{d\mathrm{P}_\omega} F(\omega_{t_1}, \ldots, \omega_{t_n})$$

- Let $(\mathrm{m}_o = \sqrt{\det \mathrm{g}}\, \mathrm{n}, \ \mathrm{m}_f := \sqrt{\det \mathrm{g}}\, \mathrm{n})$ at $t_o \leq t_f$ be assigned (as before)

scalar density: w.r.t. the invariant volume

Backward drift via Girsanov formula

- The Wiener process $\omega \equiv \{\omega_t; t \in [t_o, t_f]\}$ is invariant under time reversal.
- Semi-martingale $\xi \equiv \{\xi_t; t \in [t_o, t_f]\}$ absolutely continuous w.r.t. ω with non-vanishing covariant drift $\boldsymbol{f}$. Girsanov formula permits to write for any n-tuple $t_o \leq t_1 \leq \ldots t_n \leq t_f$

$$\mathrm{E} F(\xi_{t_1}, \ldots, \xi_{t_n}) = \mathrm{E} \frac{d\mathrm{P}_\xi}{d\mathrm{P}_\omega} F(\omega_{t_1}, \ldots, \omega_{t_n})$$

- Let $(\mathrm{m}_o = \sqrt{\det \mathrm{g}}\, \mathrm{n}, \ \mathrm{m}_f := \sqrt{\det \mathrm{g}}\, \mathrm{n})$ at $t_o \leq t_f$ be assigned (as before)

scalar density: w.r.t. the invariant volume

Ito integral w.r.t. Euclidean Wiener process

$$\frac{d\mathrm{P}_\xi}{d\mathrm{P}_\omega} = \mathrm{n}_o(\omega_{t_o})\, e^{\int_{t_o}^{t_f} \left[(\mathrm{g} \cdot \boldsymbol{f})(\boldsymbol{\omega}_t, t) \cdot (\mathrm{e} \cdot d\boldsymbol{\beta}_t) - \frac{\|\boldsymbol{f}(\boldsymbol{\omega}_t, t)\|_{\mathrm{g}}^2}{2} dt \right]}$$

post-point differential: martingale with respect to the "future" filtration

$$= \mathrm{n}_f(\omega_{t_f})\ e^{\int_{t_o}^{t_f} \left[(\mathrm{g} \cdot \tilde{\boldsymbol{f}})(\boldsymbol{\omega}_t, t) \cdot (\mathrm{e} \overset{\triangleright}{\cdot} d\boldsymbol{\beta}_t) - \frac{\|\tilde{\boldsymbol{f}}(\boldsymbol{\omega}_t, t)\|_{\mathrm{g}}^2}{2} dt \right]}$$

$$\Rightarrow \quad (\boldsymbol{f} - \tilde{\boldsymbol{f}})(\boldsymbol{x}, t) = \mathrm{g}^{-1}(\boldsymbol{x}) \cdot \partial_{\boldsymbol{x}} \ln \mathrm{n}(\boldsymbol{x}, t)$$

scalar density

A change of perspective: non-equilibrium thermodynamics of small systems

- molecular fluctuations play a fundamental role for transport processes (phase transitions, nucleation, chemical reactions, DNA mutations).
- dynamical fluctuations contrary to the thermodynamic forces are likely to occur in small systems.
- nano-scales: size of the fluctuations are of the same order of the magnitude of the observables

A change of perspective: non-equilibrium thermodynamics of small systems

- molecular fluctuations play a fundamental role for transport processes (phase transitions, nucleation, chemical reactions, DNA mutations).
- dynamical fluctuations contrary to the thermodynamic forces are likely to occur in small systems.
- nano-scales: size of the fluctuations are of the same order of the magnitude of the observables

A change of perspective: non-equilibrium thermodynamics of small systems

- molecular fluctuations play a fundamental role for transport processes (phase transitions, nucleation, chemical reactions, DNA mutations).
- dynamical fluctuations contrary to the thermodynamic forces are likely to occur in small systems.
- nano-scales: size of the fluctuations are of the same order of the magnitude of the observables

A change of perspective: fluctuation theorems

- D. J. Evans, E. G. D. Cohen, and G. P. Morriss. “Probability of second law violations in shearing steady states”. In: *Physical Review Letters* 71 (15 1993), pp. 2401–2404. DOI: 10.1103/PhysRevLett.71.2401
- G. Gallavotti and E. G. D. Cohen. “Dynamical Ensembles in Nonequilibrium Statistical Mechanics”. In: *Physical Review Letters* 74.14 (14 1995), pp. 2694–2697. DOI: 10.1103/PhysRevLett.74.2694. arXiv:chao-dyn/9410007 [nlin.CD]
- J. Kurchan. “Fluctuation theorem for stochastic dynamics”. In: *Journal of Physics A: Mathematical and General* 31.16 (1998), p. 3719. DOI: 10.1088/0305-4470/31/16/003. arXiv:cond-mat/9709304 [cond-mat.stat-mech]
- J. L. Lebowitz and H. Spohn. “A Gallavotti-Cohen Type Symmetry in the Large Deviation Functional for Stochastic Dynamics”. In: *Journal of Statistical Physics* 95.1 (1999), pp. 333–365. DOI: 10.1023/A:1004589714161. arXiv:cond-mat/9811220 [cond-mat.stat-mech]
- C. Maes, F. Redig, and A. V. Moffaert. “On the definition of entropy production, via examples”. In: *Journal of Mathematical Physics* 41.3 (2000), pp. 1528–1554. DOI: 10.1063/1.533195

A change of perspective: fluctuation theorems

- D. J. Evans, E. G. D. Cohen, and G. P. Morriss. "Probability of second law violations in shearing steady states". In: *Physical Review Letters* 71 (15 1993), pp. 2401–2404. DOI: 10.1103/PhysRevLett.71.2401
- G. Gallavotti and E. G. D. Cohen. "Dynamical Ensembles in Nonequilibrium Statistical Mechanics". In: *Physical Review Letters* 74.14 (14 1995), pp. 2694–2697. DOI: 10.1103/PhysRevLett.74.2694. arXiv:chao-dyn/9410007 [nlin.CD]
- J. Kurchan. "Fluctuation theorem for stochastic dynamics". In: *Journal of Physics A: Mathematical and General* 31.16 (1998), p. 3719. DOI: 10.1088/0305-4470/31/16/003. arXiv:cond-mat/9709304 [cond-mat.stat-mech]
- J. L. Lebowitz and H. Spohn. "A Gallavotti-Cohen Type Symmetry in the Large Deviation Functional for Stochastic Dynamics". In: *Journal of Statistical Physics* 95.1 (1999), pp. 333–365. DOI: 10.1023/A:1004589714161. arXiv:cond-mat/9811220 [cond-mat.stat-mech]
- C. Maes, F. Redig, and A. V. Moffaert. "On the definition of entropy production, via examples". In: *Journal of Mathematical Physics* 41.3 (2000), pp. 1528–1554. DOI: 10.1063/1.533195

A change of perspective: fluctuation theorems

- D. J. Evans, E. G. D. Cohen, and G. P. Morriss. “Probability of second law violations in shearing steady states”. In: *Physical Review Letters* 71 (15 1993), pp. 2401–2404. DOI: 10.1103/PhysRevLett.71.2401
- G. Gallavotti and E. G. D. Cohen. “Dynamical Ensembles in Nonequilibrium Statistical Mechanics”. In: *Physical Review Letters* 74.14 (14 1995), pp. 2694–2697. DOI: 10.1103/PhysRevLett.74.2694. arXiv:chao-dyn/9410007 [nlin.CD]
- J. Kurchan. “Fluctuation theorem for stochastic dynamics”. In: *Journal of Physics A: Mathematical and General* 31.16 (1998), p. 3719. DOI: 10.1088/0305-4470/31/16/003. arXiv:cond-mat/9709304 [cond-mat.stat-mech]
- J. L. Lebowitz and H. Spohn. “A Gallavotti-Cohen Type Symmetry in the Large Deviation Functional for Stochastic Dynamics”. In: *Journal of Statistical Physics* 95.1 (1999), pp. 333–365. DOI: 10.1023/A:1004589714161. arXiv:cond-mat/9811220 [cond-mat.stat-mech]
- C. Maes, F. Redig, and A. V. Moffaert. “On the definition of entropy production, via examples”. In: *Journal of Mathematical Physics* 41.3 (2000), pp. 1528–1554. DOI: 10.1063/1.533195

A change of perspective: fluctuation theorems

- D. J. Evans, E. G. D. Cohen, and G. P. Morriss. “Probability of second law violations in shearing steady states”. In: *Physical Review Letters* 71 (15 1993), pp. 2401–2404. DOI: 10.1103/PhysRevLett.71.2401
- G. Gallavotti and E. G. D. Cohen. “Dynamical Ensembles in Nonequilibrium Statistical Mechanics”. In: *Physical Review Letters* 74.14 (14 1995), pp. 2694–2697. DOI: 10.1103/PhysRevLett.74.2694. arXiv:chao-dyn/9410007 [nlin.CD]
- J. Kurchan. “Fluctuation theorem for stochastic dynamics”. In: *Journal of Physics A: Mathematical and General* 31.16 (1998), p. 3719. DOI: 10.1088/0305-4470/31/16/003. arXiv:cond-mat/9709304 [cond-mat.stat-mech]
- J. L. Lebowitz and H. Spohn. “A Gallavotti-Cohen Type Symmetry in the Large Deviation Functional for Stochastic Dynamics”. In: *Journal of Statistical Physics* 95.1 (1999), pp. 333–365. DOI: 10.1023/A:1004589714161. arXiv:cond-mat/9811220 [cond-mat.stat-mech]
- C. Maes, F. Redig, and A. V. Moffaert. “On the definition of entropy production, via examples”. In: *Journal of Mathematical Physics* 41.3 (2000), pp. 1528–1554. DOI: 10.1063/1.533195

A change of perspective: fluctuation theorems

- D. J. Evans, E. G. D. Cohen, and G. P. Morriss. "Probability of second law violations in shearing steady states". In: *Physical Review Letters* 71 (15 1993), pp. 2401–2404. DOI: 10.1103/PhysRevLett.71.2401
- G. Gallavotti and E. G. D. Cohen. "Dynamical Ensembles in Nonequilibrium Statistical Mechanics". In: *Physical Review Letters* 74.14 (14 1995), pp. 2694–2697. DOI: 10.1103/PhysRevLett.74.2694. arXiv:chao-dyn/9410007 [nlin.CD]
- J. Kurchan. "Fluctuation theorem for stochastic dynamics". In: *Journal of Physics A: Mathematical and General* 31.16 (1998), p. 3719. DOI: 10.1088/0305-4470/31/16/003. arXiv:cond-mat/9709304 [cond-mat.stat-mech]
- J. L. Lebowitz and H. Spohn. "A Gallavotti-Cohen Type Symmetry in the Large Deviation Functional for Stochastic Dynamics". In: *Journal of Statistical Physics* 95.1 (1999), pp. 333–365. DOI: 10.1023/A:1004589714161. arXiv:cond-mat/9811220 [cond-mat.stat-mech]
- C. Maes, F. Redig, and A. V. Moffaert. "On the definition of entropy production, via examples". In: *Journal of Mathematical Physics* 41.3 (2000), pp. 1528–1554. DOI: 10.1063/1.533195

The statistical “cost” of deterministic reversal

Girsanov martingale and time reversal

$$\int_{t_o}^{t_f} (\mathrm{g} \cdot \boldsymbol{f}) \cdot (\mathrm{e} \cdot d\beta_t) \quad \Rightarrow \int_{t_o}^{t_f} (\mathrm{g} \cdot \tilde{\boldsymbol{f}}) \cdot (\mathrm{e} \overset{\triangleright}{\cdot} d\beta_t)$$

What if we replace $\tilde{\boldsymbol{f}}$ with $-\boldsymbol{f}$?

$$\int_{t_o}^{t_f} (\mathrm{g} \cdot \boldsymbol{f}) \cdot (\mathrm{e} \overset{\triangleright}{\cdot} d\beta_t) =$$

$$2 \int_{t_o}^{t_f} (\mathrm{g} \cdot \boldsymbol{f}) \cdot (\mathrm{e} \overset{\diamond}{\cdot} d\beta_t) - \int_{t_o}^{t_f} (\mathrm{g} \cdot \boldsymbol{f}) \cdot (\mathrm{e} \overset{\triangleleft}{\cdot} d\beta_t)$$

Log-Cost of the deterministic reversal for ξ

$$\mathcal{J}[\xi] = 2 \int_{t_o}^{t_f} d\xi \overset{\diamond}{\cdot} \mathrm{g} \cdot \boldsymbol{f}$$

The statistical "cost" of deterministic reversal

Girsanov martingale and time reversal

$$\int_{t_o}^{t_f} (\mathrm{g}\cdot \boldsymbol{f})\cdot(\mathrm{e}\cdot d\beta_t) \quad \Rightarrow \int_{t_o}^{t_f} (\mathrm{g}\cdot \tilde{\boldsymbol{f}})\cdot(\mathrm{e} \overset{\triangleright}{\cdot} d\beta_t)$$

What if we replace $\tilde{\boldsymbol{f}}$ with $-\boldsymbol{f}$?

$$\int_{t_o}^{t_f} (\mathrm{g}\cdot \boldsymbol{f})\cdot(\mathrm{e} \overset{\triangleright}{\cdot} d\beta_t) =$$

$$2\int_{t_o}^{t_f} (\mathrm{g}\cdot \boldsymbol{f})\cdot(\mathrm{e} \overset{\diamond}{\cdot} d\beta_t) - \int_{t_o}^{t_f} (\mathrm{g}\cdot \boldsymbol{f})\cdot(\mathrm{e} \overset{\triangleleft}{\cdot} d\beta_t)$$

Log-Cost of the deterministic reversal for ξ

$$\mathcal{J}[\xi] = 2\int_{t_o}^{t_f} d\xi \overset{\diamond}{\cdot} \mathrm{g}\cdot \boldsymbol{f}$$

The statistical "cost" of deterministic reversal

Girsanov martingale and time reversal

$$\int_{t_o}^{t_f} (\mathrm{g} \cdot \boldsymbol{f}) \cdot (\mathrm{e} \cdot d\beta_t) \quad \Rightarrow \int_{t_o}^{t_f} (\mathrm{g} \cdot \tilde{\boldsymbol{f}}) \cdot (\mathrm{e} \stackrel{\triangleright}{\cdot} d\beta_t)$$

What if we replace $\tilde{\boldsymbol{f}}$ with $-\boldsymbol{f}$?

$$\int_{t_o}^{t_f} (\mathrm{g} \cdot \boldsymbol{f}) \cdot (\mathrm{e} \stackrel{\triangleright}{\cdot} d\beta_t) =$$

$$2 \int_{t_o}^{t_f} (\mathrm{g} \cdot \boldsymbol{f}) \cdot (\mathrm{e} \stackrel{\diamond}{\cdot} d\beta_t) - \int_{t_o}^{t_f} (\mathrm{g} \cdot \boldsymbol{f}) \cdot (\mathrm{e} \stackrel{\triangleleft}{\cdot} d\beta_t)$$

Log-Cost of the deterministic reversal for ξ

$$\mathcal{J}[\xi] = 2 \int_{t_o}^{t_f} d\xi \stackrel{\diamond}{\cdot} \mathrm{g} \cdot \boldsymbol{f}$$

Thermodynamics of over-damped dynamics

Suppose

$$\boldsymbol{f}(\boldsymbol{x}, t) = -(\mathrm{g}^{-1} \cdot \partial_{\boldsymbol{x}} U)(\boldsymbol{x}, t)$$

then

$$\mathcal{J}[\xi] = -2 \int_{t_o}^{t_f} d\xi \overset{\diamond}{\cdot} (\partial_{\xi_t} U)(\xi_t, t)$$

$$= -2[U(\xi_{t_f}, t_f) - U(\xi_{t_o}, t_o)] + 2 \int_{t_o}^{t_f} dt\, (\partial_t U)(\xi_t, t)$$

Sekimoto's interpretation

$$\mathcal{W} := \int_{t_o}^{t_f} dt\, (\partial_t U)(\xi_t, t) \qquad \text{work}$$

$$\mathcal{Q} := -\int_{t_o}^{t_f} d\xi \overset{\diamond}{\cdot} (\partial_{\xi_t} U)(\xi_t, t) \qquad \text{heat}$$

"Natural" parametrization of the Heat

Heat and development map

$$\mathcal{Q} = -\int_{t_o}^{t_f} d\xi \overset{\diamond}{\cdot} (\partial_{\xi_t} U)(\xi_t, t)$$

$$= \int_{t_o}^{t_f} \left\{ -(\mathrm{e} \cdot d\beta)_t \cdot (\partial_{\xi_t} U)(\xi_t, t) + \left(\| \mathrm{g} \cdot \partial_{\xi_t} U \|_{\mathrm{g}}^2 - \frac{1}{2} \Delta_{\xi_t} U \right) dt \right\}$$

hence

covariant density

$$\mathrm{E}\mathcal{Q} = \int_{t_o}^{t_f} \left\{ \| \mathrm{g}^{-1} \cdot \partial_{\xi_t} U \|_{\mathrm{g}}^2 + \frac{1}{2} (\partial_{\xi_t} \ln \mathrm{n}) \cdot \mathrm{g}^{-1} \cdot (\partial_{\xi_t} U) \right\} dt$$

"Natural" parametrization of the Heat

Heat and development map

$$\mathcal{Q} = -\int_{t_o}^{t_f} d\xi \overset{\diamond}{\cdot} (\partial_{\xi_t} U)(\xi_t, t)$$

$de = -\Gamma : e \otimes d\xi$

vanishes on average

$$= \int_{t_o}^{t_f} \left\{ -(e \cdot d\beta)_t \cdot (\partial_{\xi_t} U)(\xi_t, t) + \left(\| g \cdot \partial_{\xi_t} U \|_g^2 - \frac{1}{2} \Delta_{\xi_t} U \right) dt \right\}$$

hence

covariant density

$$\mathrm{E}\mathcal{Q} = \int_{t_o}^{t_f} \left\{ \| g^{-1} \cdot \partial_{\xi_t} U \|_g^2 + \frac{1}{2} (\partial_{\xi_t} \ln \mathrm{n}) \cdot g^{-1} \cdot (\partial_{\xi_t} U) \right\} dt$$

“Natural” parametrization of the Heat

Heat and development map

$$\mathcal{Q} = -\int_{t_o}^{t_f} d\xi \overset{\diamond}{\cdot} (\partial_{\xi_t} U)(\xi_t, t)$$

$de = -\Gamma : e \otimes d\xi$

vanishes on average

$$= \int_{t_o}^{t_f} \left\{ -(e \cdot d\beta)_t \cdot (\partial_{\xi_t} U)(\xi_t, t) + \left(\| g \cdot \partial_{\xi_t} U \|_g^2 - \frac{1}{2} \Delta_{\xi_t} U \right) dt \right\}$$

hence

covariant density

$$\mathrm{E}\mathcal{Q} = \int_{t_o}^{t_f} \left\{ \| g^{-1} \cdot \partial_{\xi_t} U \|_g^2 + \frac{1}{2} (\partial_{\xi_t} \ln \mathrm{n}) \cdot g^{-1} \cdot (\partial_{\xi_t} U) \right\} dt$$

"Natural" parametrization of the Heat

Nelson's current and osmotic velocity

$$\boldsymbol{v} = \frac{\boldsymbol{f} + \tilde{\boldsymbol{f}}}{2} \quad \& \quad \boldsymbol{u} = \frac{\boldsymbol{f} - \tilde{\boldsymbol{f}}}{2} = \frac{1}{2} \mathrm{g}^{-1} \partial_{\boldsymbol{x}} \ln \mathrm{n}$$

$$\partial_t \mathrm{n} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v}\, \mathrm{n}) = 0$$

- current $\boldsymbol{v}$ and osmotic velocities $\boldsymbol{u}$ transform as vector fields
- $\boldsymbol{v}$ behaves as the velocity field of a deterministic ensemble.

The "natural" representation of the Heat

$$\mathrm{E}\mathcal{Q} = \mathrm{E} \ln \mathrm{n}(\boldsymbol{x}_t, t)\big|_{t_o}^{t_f} + \mathrm{E} \int_{t_o}^{t_f} dt \parallel \boldsymbol{v} \parallel_{\mathrm{g}}^2$$

“Natural” parametrization of the Heat

Nelson’s current and osmotic velocity

$$\boldsymbol{v} = \frac{\boldsymbol{f} + \tilde{\boldsymbol{f}}}{2} \quad \& \quad \boldsymbol{u} = \frac{\boldsymbol{f} - \tilde{\boldsymbol{f}}}{2} = \frac{1}{2}\mathrm{g}^{-1}\partial_{\boldsymbol{x}} \ln \mathrm{n}$$

$$\partial_t \mathrm{n} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v}\,\mathrm{n}) = 0$$

- current $\boldsymbol{v}$ and osmotic velocities $\boldsymbol{u}$ transform as vector fields
- $\boldsymbol{v}$ behaves as the velocity field of a deterministic ensemble.

The “natural” representation of the Heat

$$\mathrm{E}Q = \mathrm{E} \ln \mathrm{n}(\boldsymbol{x}_t, t)\big|_{t_o}^{t_f} + \mathrm{E}\int_{t_o}^{t_f} dt \, \parallel \boldsymbol{v} \parallel_{\mathrm{g}}^2$$

"Natural" parametrization of the Heat

Nelson's current and osmotic velocity

$$\boldsymbol{v} = \frac{\boldsymbol{f} + \tilde{\boldsymbol{f}}}{2} \quad \& \quad \boldsymbol{u} = \frac{\boldsymbol{f} - \tilde{\boldsymbol{f}}}{2} = \frac{1}{2}\mathrm{g}^{-1}\partial_{\boldsymbol{x}} \ln \mathrm{n}$$

$$\partial_t \mathrm{n} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v}\,\mathrm{n}) = 0$$

- current $\boldsymbol{v}$ and osmotic velocities $\boldsymbol{u}$ transform as vector fields
- $\boldsymbol{v}$ behaves as the velocity field of a deterministic ensemble.

The "natural" representation of the Heat

$$\mathrm{E}\mathcal{Q} = \mathrm{E} \ln \mathrm{n}(\boldsymbol{x}_t, t)\Big|_{t_o}^{t_f} + \mathrm{E}\int_{t_o}^{t_f} dt \parallel \boldsymbol{v} \parallel_{\mathrm{g}}^2$$

"Natural" parametrization of the Heat

Nelson's current and osmotic velocity

$$\boldsymbol{v} = \frac{\boldsymbol{f} + \tilde{\boldsymbol{f}}}{2} \quad \& \quad \boldsymbol{u} = \frac{\boldsymbol{f} - \tilde{\boldsymbol{f}}}{2} = \frac{1}{2}\mathsf{g}^{-1}\partial_{\boldsymbol{x}} \ln \mathrm{n}$$

$$\partial_t \mathrm{n} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v}\,\mathrm{n}) = 0$$

- current $\boldsymbol{v}$ and osmotic velocities $\boldsymbol{u}$ transform as vector fields
- $\boldsymbol{v}$ behaves as the velocity field of a deterministic ensemble.

The "natural" representation of the Heat

$$\mathrm{E}\mathcal{Q} = \mathrm{E} \ln \mathrm{n}(\boldsymbol{x}_t, t)\big|_{t_o}^{t_f} + \mathrm{E}\int_{t_o}^{t_f} dt \parallel \boldsymbol{v} \parallel_{\mathsf{g}}^2$$

Bounds and coercivity ($\beta = 1$)

Bound for the Heat

$$\mathrm{E}\mathcal{Q} \geq \mathrm{E} \ln \mathrm{n}(\boldsymbol{x}_t, t)|_{t_o}^{t_f} = -\text{variation of Shannon-Gibbs Entropy}$$

Bound for the work

$$\mathrm{E}\mathcal{W} = \mathrm{E}\left\{ U(\boldsymbol{x}_t, t)|_{t_o}^{t_f} + \mathcal{Q} \right\} \geq$$

$$\mathrm{E}\left\{ U(\boldsymbol{x}_t, t) + \frac{1}{2} \ln \mathrm{n}(\boldsymbol{x}_t, t) \right\}|_{t_o}^{t_f} = \text{variation of equilibrium free energy}$$

Current velocity: deviation from equilibrium

$$\boldsymbol{v} = -\mathrm{g}^{-1} \cdot \partial_{\boldsymbol{x}} \left(U + \frac{1}{2} \ln n \right) \qquad \text{vanishes at equilibrium !!!}$$

Coercivity

The Heat release between assigned end state is a convex functional of the current velocity.

Bounds and coercivity ($\beta = 1$)

Bound for the Heat

$$\mathrm{E}\mathcal{Q} \geq \mathrm{E}\ln \mathrm{n}(\boldsymbol{x}_t, t)|_{t_o}^{t_f} = -\text{variation of Shannon-Gibbs Entropy}$$

Bound for the work

$$\mathrm{E}\mathcal{W} = \mathrm{E}\left\{U(\boldsymbol{x}_t, t)|_{t_o}^{t_f} + \mathcal{Q}\right\} \geq$$

$$\mathrm{E}\left\{U(\boldsymbol{x}_t, t) + \frac{1}{2}\ln \mathrm{n}(\boldsymbol{x}_t, t)\right\}|_{t_o}^{t_f} = \text{variation of equilibrium free energy}$$

Current velocity: deviation from equilibrium

$$\boldsymbol{v} = -\mathrm{g}^{-1} \cdot \partial_{\boldsymbol{x}} \left(U + \frac{1}{2}\ln n\right) \qquad \text{vanishes at equilibrium !!!}$$

Coercivity

The Heat release between assigned end state is a convex functional of the current velocity.

Bounds and coercivity ($\beta = 1$)

Bound for the Heat

$$\mathrm{E}\mathcal{Q} \geq \mathrm{E} \ln \mathrm{n}(\boldsymbol{x}_t, t)|_{t_o}^{t_f} = -\text{variation of Shannon-Gibbs Entropy}$$

Bound for the work

$$\mathrm{E}\mathcal{W} = \mathrm{E}\left\{ U(\boldsymbol{x}_t, t)|_{t_o}^{t_f} + \mathcal{Q} \right\} \geq$$

$$\mathrm{E}\left\{ U(\boldsymbol{x}_t, t) + \frac{1}{2} \ln \mathrm{n}(\boldsymbol{x}_t, t) \right\}|_{t_o}^{t_f} = \text{variation of equilibrium free energy}$$

Current velocity: deviation from equilibrium

$$\boldsymbol{v} = -\mathrm{g}^{-1} \cdot \partial_{\boldsymbol{x}} \left(U + \frac{1}{2} \ln n \right) \qquad \text{vanishes at equilibrium !!!}$$

Coercivity

The Heat release between assigned end state is a convex functional of the current velocity.

Bounds and coercivity ($\beta = 1$)

Bound for the Heat

$$\mathrm{E}\mathcal{Q} \geq \mathrm{E}\ln \mathrm{n}(\boldsymbol{x}_t, t)|_{t_o}^{t_f} = -\text{variation of Shannon-Gibbs Entropy}$$

Bound for the work

$$\mathrm{E}\mathcal{W} = \mathrm{E}\left\{U(\boldsymbol{x}_t, t)|_{t_o}^{t_f} + \mathcal{Q}\right\} \geq$$

$$\mathrm{E}\left\{U(\boldsymbol{x}_t, t) + \frac{1}{2}\ln \mathrm{n}(\boldsymbol{x}_t, t)\right\}|_{t_o}^{t_f} = \text{variation of equilibrium free energy}$$

Current velocity: deviation from equilibrium

$$\boldsymbol{v} = -\mathrm{g}^{-1} \cdot \partial_{\boldsymbol{x}} \left(U + \frac{1}{2}\ln n\right) \qquad \text{vanishes at equilibrium !!!}$$

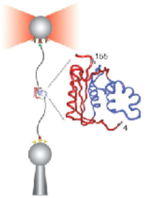
Coercivity

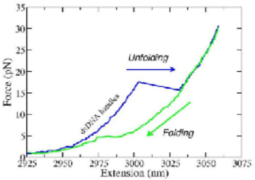
The Heat release between assigned end state is a convex functional of the current velocity.

Physical motivations for control

Nano motors

- mainly in steady operation
- what external control should be apply that maximizes the output power?
- operation under minimal dissipation
- efficiency at maximum power



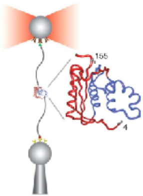


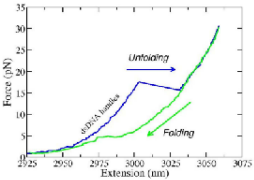
A. Alemany, M. Ribezzi, and F. Ritort. "Recent progress in fluctuation theorems and free energy recovery". In: *AIP Conference Proceedings* 1332.1 (2011). Ed. by P. L. Garrido, J. Marro, and F. de los Santos, pp. 96–110. DOI: 10.1063/1.3569489. arXiv:1101.3174 [cond-mat.stat-mech]

Physical motivations for control

Nano motors

- mainly in steady operation
- what external control should be apply that maximizes the output power?
- operation under minimal dissipation
- efficiency at maximum power



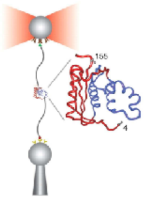


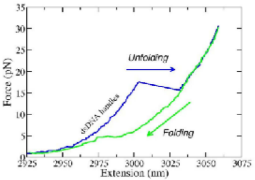
A. Alemany, M. Ribezzi, and F. Ritort. "Recent progress in fluctuation theorems and free energy recovery". In: *AIP Conference Proceedings* 1332.1 (2011). Ed. by P. L. Garrido, J. Marro, and F. de los Santos, pp. 96–110. DOI: 10.1063/1.3569489. arXiv:1101.3174 [cond-mat.stat-mech]

Physical motivations for control

Nano motors

- mainly in steady operation
- what external control should be apply that maximizes the output power?
- operation under minimal dissipation
- efficiency at maximum power





A. Alemany, M. Ribezzi, and F. Ritort. “Recent progress in fluctuation theorems and free energy recovery”. In: *AIP Conference Proceedings* 1332.1 (2011). Ed. by P. L. Garrido, J. Marro, and F. de los Santos, pp. 96–110. DOI: 10.1063/1.3569489. arXiv:1101.3174 [cond-mat.stat-mech]

Relation to hydrodynamical optimal transport I

- Heat minimization between fixed end-states $(\mathrm{n}_o\,,\mathrm{n}_f)$ reduces to the deterministic control of

$$\mathrm{E}\,\mathcal{S} = \mathrm{E}\int_{t_o}^{t_f} dt\ \parallel \boldsymbol{v} \parallel_{\mathrm{g}}^2$$

- The problem is well-posed in the space of smooth diffusions.
- The bound lower becomes tight if jump processes are admissible minimizers:
  - no penalty on acceleration.
  - jump between equilibria: current velocity identically vanishing.

Minimization in the space of smooth diffusions

- Suppose now g = I
- $\boldsymbol{v} = \partial_{\boldsymbol{x}}\varphi$

Relation to hydrodynamical optimal transport I

- Heat minimization between fixed end-states $(\mathrm{n}_o\,,\mathrm{n}_f)$ reduces to the deterministic control of

$$\mathrm{E}\,\mathcal{S} = \mathrm{E}\int_{t_o}^{t_f} dt\ \parallel \boldsymbol{v} \parallel_{\mathrm{g}}^2$$

- The problem is well-posed in the space of smooth diffusions.
- The bound lower becomes tight if jump processes are admissible minimizers:
  - no penalty on acceleration.
  - jump between equilibria: current velocity identically vanishing.

Minimization in the space of smooth diffusions

- Suppose now $\mathrm{g} = \mathrm{I}$
- $\boldsymbol{v} = \partial_{\boldsymbol{x}}\varphi$

Relation to hydrodynamical optimal transport I

- Heat minimization between fixed end-states $(\mathrm{n}_o\,,\mathrm{n}_f)$ reduces to the deterministic control of

$$\mathrm{E}\,\mathcal{S} = \mathrm{E}\int_{t_o}^{t_f} dt\ \parallel \boldsymbol{v} \parallel_{\mathrm{g}}^2$$

- The problem is well-posed in the space of smooth diffusions.
- The bound lower becomes tight if jump processes are admissible minimizers:
  - no penalty on acceleration.
  - jump between equilibria: current velocity identically vanishing.

Minimization in the space of smooth diffusions

- Suppose now $\mathrm{g} = \mathrm{I}$
- $\boldsymbol{v} = \partial_{\boldsymbol{x}}\varphi$

Relation to hydrodynamical optimal transport I

- Heat minimization between fixed end-states $(\mathrm{n}_o\,,\mathrm{n}_f)$ reduces to the deterministic control of

$$\mathrm{E}\,\mathcal{S} = \mathrm{E}\int_{t_o}^{t_f} dt\ \parallel \boldsymbol{v} \parallel_{\mathrm{g}}^{2}$$

- The problem is well-posed in the space of smooth diffusions.
- The bound lower becomes tight if jump processes are admissible minimizers:
  - no penalty on acceleration.
  - jump between equilibria: current velocity identically vanishing.

Minimization in the space of smooth diffusions

- Suppose now g = I
- $\boldsymbol{v} = \partial_{\boldsymbol{x}}\varphi$

Relation to hydrodynamical optimal transport I

- Heat minimization between fixed end-states $(\mathrm{n}_o\,,\mathrm{n}_f)$ reduces to the deterministic control of

$$\mathrm{E}\,\mathcal{S} = \mathrm{E}\int_{t_o}^{t_f} dt \;\parallel \boldsymbol{v} \parallel_{\mathrm{g}}^2$$

- The problem is well-posed in the space of smooth diffusions.
- The bound lower becomes tight if jump processes are admissible minimizers:
  - no penalty on acceleration.
  - jump between equilibria: current velocity identically vanishing.

Minimization in the space of smooth diffusions

- Suppose now $\mathrm{g} = \mathrm{I}$
- $\boldsymbol{v} = \partial_{\boldsymbol{x}}\varphi$

Relation to hydrodynamical optimal transport I

- Heat minimization between fixed end-states $(\mathrm{n}_o\,,\mathrm{n}_f)$ reduces to the deterministic control of

$$\mathrm{E}\,\mathcal{S} = \mathrm{E}\int_{t_o}^{t_f} dt\ \parallel \boldsymbol{v} \parallel_{\mathrm{g}}^{2}$$

- The problem is well-posed in the space of smooth diffusions.
- The bound lower becomes tight if jump processes are admissible minimizers:
  - no penalty on acceleration.
  - jump between equilibria: current velocity identically vanishing.

Minimization in the space of smooth diffusions

- Suppose now $\mathrm{g} = \mathrm{I}$
- $\boldsymbol{v} = \partial_{\boldsymbol{x}}\varphi$

Relation to hydrodynamical optimal transport I

- Heat minimization between fixed end-states $(\mathrm{n}_o\,,\mathrm{n}_f)$ reduces to the deterministic control of

$$\mathrm{E}\,\mathcal{S} = \mathrm{E}\int_{t_o}^{t_f} dt \;\parallel \boldsymbol{v} \parallel_{\mathrm{g}}^2$$

- The problem is well-posed in the space of smooth diffusions.
- The bound lower becomes tight if jump processes are admissible minimizers:
  - no penalty on acceleration.
  - jump between equilibria: current velocity identically vanishing.

Minimization in the space of smooth diffusions

- Suppose now $\mathrm{g} = \mathrm{I}$
- $\boldsymbol{v} = \partial_{\boldsymbol{x}}\varphi$

Relation to hydrodynamical optimal transport I

- Heat minimization between fixed end-states $(\mathrm{n}_o\,,\mathrm{n}_f)$ reduces to the deterministic control of

$$\mathrm{E}\,\mathcal{S} = \mathrm{E}\int_{t_o}^{t_f} dt \;\parallel \boldsymbol{v} \parallel_{\mathrm{g}}^{2}$$

- The problem is well-posed in the space of smooth diffusions.
- The bound lower becomes tight if jump processes are admissible minimizers:
  - no penalty on acceleration.
  - jump between equilibria: current velocity identically vanishing.

Minimization in the space of smooth diffusions

- Suppose now $\mathrm{g} = \mathrm{I}$
- $\boldsymbol{v} = \partial_{\boldsymbol{x}}\varphi$

$$\partial_t\varphi + \frac{1}{2}\parallel \partial_{\boldsymbol{x}}\varphi \parallel^2 = 0$$

$$\partial_t \mathrm{m} + \partial_{\boldsymbol{x}}\cdot(\mathrm{m}\,\partial_{\boldsymbol{x}}\varphi) = 0$$

$$\mathrm{m}(\boldsymbol{x}, t_o) = \mathrm{m}_o(\boldsymbol{x}) \quad \& \quad \mathrm{m}(\boldsymbol{x}, t_f) = \mathrm{m}_f(\boldsymbol{x})$$

Relation to hydrodynamical optimal transport II

- Existence and uniqueness of solutions in viscosity sense, Burgers dynamics.
- Burgers equation with initial and final densities is a well studied problem.
- Example: Monge-Ampére-Kantorovich reconstruction of the mass density early Universe

Relation to hydrodynamical optimal transport II

- Existence and uniqueness of solutions in viscosity sense, Burgers dynamics.
- Burgers equation with initial and final densities is a well studied problem.
- Example: Monge-Ampére-Kantorovich reconstruction of the mass density early Universe

Relation to hydrodynamical optimal transport II

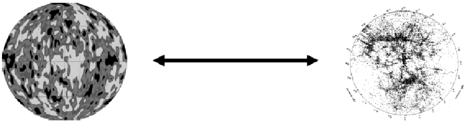
- Existence and uniqueness of solutions in viscosity sense, Burgers dynamics.
- Burgers equation with initial and final densities is a well studied problem.
- Example: Monge-Ampére-Kantorovich reconstruction of the mass density early Universe

Frisch et al, Nature **417**, 260 (2002)

Brenier et al, MNRAS **346**, 501 (2003)

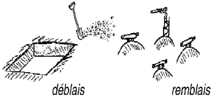
Relation to hydrodynamical optimal transport II

- Existence and uniqueness of solutions in viscosity sense, Burgers dynamics.
- Burgers equation with initial and final densities is a well studied problem.
- Example: Monge-Ampére-Kantorovich reconstruction of the mass density early Universe



Frisch et al, Nature **417**, 260 (2002)

Brenier et al, MNRAS **346**, 501 (2003)



Villani, Topics in Optimal Transportation (AMS 2003)

A "valley view" over optimal thermodynamic control

Ideas borrowed from instanton calculus in Quantum Field Theory.

$$\mathcal{A} := \int_{t_o}^{t_f} dt \, \left(\| \boldsymbol{v}_t \|^2 + \varepsilon \, \| \boldsymbol{a}_t \|^2 \right) + \boldsymbol{\lambda} \cdot \int_{t_o}^{t_f} dt \left[\boldsymbol{v}_t - \frac{\phi(\boldsymbol{x}_o) - \boldsymbol{x}_o}{\Delta t} \right]$$

- $\dot{\boldsymbol{x}}_t = \boldsymbol{v}_t$
- λ enforces $\boldsymbol{x}_f = \phi(\boldsymbol{x}_o)$
- ϕ relates the initial and final states

$$\mathrm{m}_f(\phi(\boldsymbol{x})) \left| \det \frac{\partial \phi(\boldsymbol{x})}{\partial \boldsymbol{x}} \right| = \mathrm{m}_o(\boldsymbol{x})$$

- It is possible to impose boundary conditions on the state $\boldsymbol{x}_{t_o} = \boldsymbol{x}_o$, $\boldsymbol{x}_{t_f} = \boldsymbol{x}_f$ and on the initial and final velocities.

A “valley view” over optimal thermodynamic control

Ideas borrowed from instanton calculus in Quantum Field Theory.

$$\mathcal{A} := \int_{t_o}^{t_f} dt \, \left(\| \boldsymbol{v}_t \|^2 + \varepsilon \, \| \boldsymbol{a}_t \|^2 \right) + \boldsymbol{\lambda} \cdot \int_{t_o}^{t_f} dt \left[\boldsymbol{v}_t - \frac{\phi(\boldsymbol{x}_o) - \boldsymbol{x}_o}{\Delta t} \right]$$

- $\dot{\boldsymbol{x}}_t = \boldsymbol{v}_t$
- $\boldsymbol{\lambda}$ enforces $\boldsymbol{x}_f = \phi(\boldsymbol{x}_o)$
- ϕ relates the initial and final states

$$\mathrm{m}_f(\phi(\boldsymbol{x})) \left| \det \frac{\partial \phi(\boldsymbol{x})}{\partial \boldsymbol{x}} \right| = \mathrm{m}_o(\boldsymbol{x})$$

- It is possible to impose boundary conditions on the state $\boldsymbol{x}_{t_o} = \boldsymbol{x}_o$, $\boldsymbol{x}_{t_f} = \boldsymbol{x}_f$ and on the initial and final velocities.

A “valley view” over optimal thermodynamic control

Ideas borrowed from instanton calculus in Quantum Field Theory.

$$\mathcal{A} := \int_{t_o}^{t_f} dt\ \left(\| \boldsymbol{v}_t \|^2 + \varepsilon\ \| \boldsymbol{a}_t \|^2\right) + \lambda \cdot \int_{t_o}^{t_f} dt \left[\boldsymbol{v}_t - \frac{\phi(\boldsymbol{x}_o) - \boldsymbol{x}_o}{\Delta t} \right]$$

- $\dot{\boldsymbol{x}}_t = \boldsymbol{v}_t$
- λ enforces $\boldsymbol{x}_f = \phi(\boldsymbol{x}_o)$
- ϕ relates the initial and final states

$$\mathrm{m}_f(\phi(\boldsymbol{x})) \left| \det \frac{\partial \phi(\boldsymbol{x})}{\partial \boldsymbol{x}} \right| = \mathrm{m}_o(\boldsymbol{x})$$

- It is possible to impose boundary conditions on the state $\boldsymbol{x}_{t_o} = \boldsymbol{x}_o$, $\boldsymbol{x}_{t_f} = \boldsymbol{x}_f$ and on the initial and final velocities.

A "valley view" over optimal thermodynamic control

Ideas borrowed from instanton calculus in Quantum Field Theory.

$$\mathcal{A} := \int_{t_o}^{t_f} dt \, \left(\| \boldsymbol{v}_t \|^2 + \varepsilon \, \| \boldsymbol{a}_t \|^2 \right) + \lambda \cdot \int_{t_o}^{t_f} dt \left[\boldsymbol{v}_t - \frac{\phi(\boldsymbol{x}_o) - \boldsymbol{x}_o}{\Delta t} \right]$$

- $\dot{\boldsymbol{x}}_t = \boldsymbol{v}_t$
- λ enforces $\boldsymbol{x}_f = \phi(\boldsymbol{x}_o)$
- ϕ relates the initial and final states

$$\mathrm{m}_f(\phi(\boldsymbol{x})) \left| \det \frac{\partial \phi(\boldsymbol{x})}{\partial \boldsymbol{x}} \right| = \mathrm{m}_o(\boldsymbol{x})$$

- It is possible to impose boundary conditions on the state $\boldsymbol{x}_{t_o} = \boldsymbol{x}_o$, $\boldsymbol{x}_{t_f} = \boldsymbol{x}_f$ and on the initial and final velocities.

More about all the above

- E. Aurell, C. Mejía-Monasterio, and P. Muratore-Ginanneschi. "Optimal protocols and optimal transport in stochastic thermodynamics". In: *Physical Review Letters* 106.25 (2011), p. 250601. DOI: 10.1103/PhysRevLett.106.250601. arXiv:1012.2037 [cond-mat.stat-mech]
- E. Aurell, C. Mejía-Monasterio, and P. Muratore-Ginanneschi. "Boundary layers in stochastic thermodynamics". In: *Physical Review E* 85.2 (2 2012), 020103(R). DOI: 10.1103/PhysRevE.85.020103. arXiv:1111.2876 [cond-mat.stat-mech]
- E. Aurell et al. "Refined Second Law of Thermodynamics for fast random processes". In: *Journal of Statistical Physics* 147.3 (2012), pp. 487–505. DOI: 10.1007/s10955-012-0478-x. arXiv:1201.3207 [cond-mat.stat-mech]
- P. Muratore-Ginanneschi, C. Mejía-Monasterio, and L. Peliti. "Heat release by controlled continuous-time Markov jump processes". In: *ArXiv Nonlinear Sciences e-prints* (2012). arXiv:1203.4062 [cond-mat.stat-mech]

Thanks to

- Erik Aurell (KTH and Aalto University)
- Krzysztof Gawędzki (ENS, Lyon)
- Carlos Carlos Mejía-Monasterio (Technical University of Madrid)
- Roya Mohayaee (CNRS, Paris)
- Luca Peliti (University of Napoli)

Special thanks to

- F. Guerra and L. M. Morato. “Quantization of dynamical systems and stochastic control theory”. In: *Physical Review D* 27.8 (1983), pp. 1774–1786. DOI: 10.1103/PhysRevD.27.1774

the reading whereof provided the original inspiration for my work on this topic.

Thanks to

- Erik Aurell (KTH and Aalto University)
- Krzysztof Gawędzki (ENS, Lyon)
- Carlos Carlos Mejía-Monasterio (Technical University of Madrid)
- Roya Mohayaee (CNRS, Paris)
- Luca Peliti (University of Napoli)

Special thanks to

- F. Guerra and L. M. Morato. "Quantization of dynamical systems and stochastic control theory". In: *Physical Review D* 27.8 (1983), pp. 1774–1786. DOI: 10.1103/PhysRevD.27.1774

the reading whereof provided the original inspiration for my work on this topic.

Thanks to

- Erik Aurell (KTH and Aalto University)
- Krzysztof Gawędzki (ENS, Lyon)
- Carlos Carlos Mejía-Monasterio (Technical University of Madrid)
- Roya Mohayaee (CNRS, Paris)
- Luca Peliti (University of Napoli)

Special thanks to

- F. Guerra and L. M. Morato. "Quantization of dynamical systems and stochastic control theory". In: *Physical Review D* 27.8 (1983), pp. 1774–1786. DOI: 10.1103/PhysRevD.27.1774

the reading whereof provided the original inspiration for my work on this topic.

Thanks to

- Erik Aurell (KTH and Aalto University)
- Krzysztof Gawędzki (ENS, Lyon)
- Carlos Carlos Mejía-Monasterio (Technical University of Madrid)
- Roya Mohayaee (CNRS, Paris)
- Luca Peliti (University of Napoli)

Special thanks to

- F. Guerra and L. M. Morato. “Quantization of dynamical systems and stochastic control theory”. In: *Physical Review D* 27.8 (1983), pp. 1774–1786. DOI: 10.1103/PhysRevD.27.1774

the reading whereof provided the original inspiration for my work on this topic.

Thanks to

- Erik Aurell (KTH and Aalto University)
- Krzysztof Gawędzki (ENS, Lyon)
- Carlos Carlos Mejía-Monasterio (Technical University of Madrid)
- Roya Mohayaee (CNRS, Paris)
- Luca Peliti (University of Napoli)

Special thanks to

- F. Guerra and L. M. Morato. “Quantization of dynamical systems and stochastic control theory”. In: *Physical Review D* 27.8 (1983), pp. 1774–1786. DOI: 10.1103/PhysRevD.27.1774

the reading whereof provided the original inspiration for my work on this topic.

Thanks to

- Erik Aurell (KTH and Aalto University)
- Krzysztof Gawędzki (ENS, Lyon)
- Carlos Carlos Mejía-Monasterio (Technical University of Madrid)
- Roya Mohayaee (CNRS, Paris)
- Luca Peliti (University of Napoli)

Special thanks to

- F. Guerra and L. M. Morato. "Quantization of dynamical systems and stochastic control theory". In: *Physical Review D* 27.8 (1983), pp. 1774–1786. DOI: 10.1103/PhysRevD.27.1774

the reading whereof provided the original inspiration for my work on this topic.

A. Alemany, M. Ribezzi, and F. Ritort. "Recent progress in fluctuation theorems and free energy recovery". In: *AIP Conference Proceedings* 1332.1 (2011). Ed. by P. L. Garrido, J. Marro, and F. de los Santos, pp. 96–110. DOI: 10.1063/1.3569489. arXiv:1101.3174 [cond-mat.stat-mech].

E. Aurell, C. Mejía-Monasterio, and P. Muratore-Ginanneschi. "Boundary layers in stochastic thermodynamics". In: *Physical Review E* 85.2 (2 2012), 020103(R). DOI: 10.1103/PhysRevE.85.020103. arXiv:1111.2876 [cond-mat.stat-mech].

E. Aurell, C. Mejía-Monasterio, and P. Muratore-Ginanneschi. "Optimal protocols and optimal transport in stochastic thermodynamics". In: *Physical Review Letters* 106.25 (2011), p. 250601. DOI: 10.1103/PhysRevLett.106.250601. arXiv:1012.2037 [cond-mat.stat-mech].

E. Aurell et al. "Refined Second Law of Thermodynamics for fast random processes". In: *Journal of Statistical Physics* 147.3 (2012), pp. 487–505. DOI: 10.1007/s10955-012-0478-x. arXiv:1201.3207 [cond-mat.stat-mech].

S. N. Bernstein. “Sur les liaisons entre les grandeurs aleatoires”. In: *Proceedings of the International Congress of Mathematicians, Zürich* I (1932), pp. 288–309.

P. Dai Pra. “A stochastic control approach to reciprocal diffusion processes”. In: *Applied Mathematics and Optimization* 23.1 (1991), pp. 313–329. DOI: 10.1007/BF01442404.

J. Eells and K. D. Elworthy. “On Fredholm manifolds”. In: *Acts of the International Congress of Mathematicians,* II (1970), pp. 215–220.

D. J. Evans, E. G. D. Cohen, and G. P. Morriss. “Probability of second law violations in shearing steady states”. In: *Physical Review Letters* 71 (15 1993), pp. 2401–2404. DOI: 10.1103/PhysRevLett.71.2401.

G. Gallavotti and E. G. D. Cohen. “Dynamical Ensembles in Nonequilibrium Statistical Mechanics”. In: *Physical Review Letters* 74.14 (14 1995), pp. 2694–2697. DOI: 10.1103/PhysRevLett.74.2694. arXiv:chao-dyn/9410007 [nlin.CD].

F. Guerra and L. M. Morato. “Quantization of dynamical systems and stochastic control theory”. In: *Physical Review D* 27.8 (1983), pp. 1774–1786. DOI: 10.1103/PhysRevD.27.1774.

B. Jamison. "Reciprocal processes". In: *Probability Theory and Related Fields* 30.1 (1974), pp. 65–86. DOI: 10.1007/BF00532864.

A. N. Kolmogorov. "Zur Theorie der Markoffschen Ketten". In: *Mathematische Annalen* 112.1 (1936), pp. 155–160. DOI: 10.1007/BF01565412.

A. N. Kolmogorov. "Zur Umkehrbarkeit der statistischen Naturgesetze". In: *Mathematische Annalen* 113 (1 1937), pp. 766–772. DOI: 10.1007/BF01571664.

A. J. Krener. "Reciprocal diffusions in flat space". In: *Probability Theory and Related Fields* 107 (2 1997), pp. 243–281. DOI: 10.1007/s004400050085.

J. Kurchan. "Fluctuation theorem for stochastic dynamics". In: *Journal of Physics A: Mathematical and General* 31.16 (1998), p. 3719. DOI: 10.1088/0305-4470/31/16/003. arXiv:cond-mat/9709304 [cond-mat.stat-mech].

J. L. Lebowitz and H. Spohn. "A Gallavotti-Cohen Type Symmetry in the Large Deviation Functional for Stochastic Dynamics". In: *Journal of Statistical Physics* 95.1 (1999), pp. 333–365. DOI:

10.1023/A:1004589714161. arXiv:cond-mat/9811220 [cond-mat.stat-mech].

C. Maes, F. Redig, and A. V. Moffaert. "On the definition of entropy production, via examples". In: *Journal of Mathematical Physics* 41.3 (2000), pp. 1528–1554. DOI: 10.1063/1.533195.

P. Muratore-Ginanneschi, C. Mejía-Monasterio, and L. Peliti. "Heat release by controlled continuous-time Markov jump processes". In: *ArXiv Nonlinear Sciences e-prints* (2012). arXiv:1203.4062 [cond-mat.stat-mech].

E. Nelson. *Dynamical Theories of Brownian Motion*. second edition. Princeton University Press, 2001, p. 148.

E. Schrödinger. "Über die Umkehrung der Naturgesetze". In: *Sitzungsberichte der preussischen Akademie der Wissenschaften, physikalische mathematische Klasse* 8.9 (1931), pp. 144–153. DOI: 10.1002/ange.19310443014.