Gutzwiller's trace formula and functional determinants

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Density of quantum states in the semiclassical limit

$$\rho(E) = \lim_{\Im z \downarrow 0} \int_0^\infty \frac{dT}{\pi i} e^{\frac{izT}{\hbar}} Tr\,\hat{K}(T)$$

$\Re z = E$ Energy

$$Tr\,\hat{K}(T) := \int dx\, K(x, t+T | x, t)$$

Gutzwiller trace formula

$$\rho(E) \sim \int \frac{dp\,dq}{(2\pi\hbar)^d}\,\delta(E \quad H(p,q))$$

$$+\Im\,\frac{i}{\pi\hbar}\sum_{p,r}\frac{T_p\,e^{\frac{i\,r\,S_p(E)}{\hbar}\quad i\frac{\pi}{2}\mu_{p,r}}}{det_{\perp}(1 \quad M_{p,r})}$$

p,r Sum over periodic orbits and their repetitions

$\mu_{p.r}$ Maslov index: topological stability of the orbit

$M_{p,0}$ Monodromy matrix: only transversal fluctuations

Path integral formalism

Integration over the loop space
(by analytic continuation from the Wiener measure)

$$Tr\,\hat{K}(T) = \int_{q(t+T)=q(t)} D[q(t)]\, e^{\frac{iS}{\hbar}}$$

In the neighborhood of classical trajectories

$$\delta^2 S = \delta q (\partial_{\dot{q}} L)_{q_{cl}}\Big|_t^{T+t} + \frac{\hbar}{2}\int_t^{T+t} ds\, \delta q\, D^2_{q_{cl}} L\, \delta q + o(\hbar)$$

The boundary form is zero if the classical trajectory belongs
to the loop space

Mathematical definition of the functional determinant

For finite dimensional matrices:

$$\log Det\, O = \left[\frac{d}{ds} Tr\, O^{\ s} \right]_{s=0}$$

For infinite dimensional operators $Tr\, O^{\ s}$ converges for $\Re\, s$ large enough. Theorems by Seeley insure analytical continuation to a meromorphic function in the complex plane homomorphic in $s=0$

$$O^{\ s} = \frac{1}{2\pi i} \int_\gamma d\lambda \frac{\lambda^{\ s}}{O \quad \lambda}$$

Families of trace class operators

Let O_u be a one-parameter u family of elliptic operators on a d-dimensional manifold M with (d-1)-dimensional boundary where O_u satisfies some boundary conditions B. Then

$$\frac{d}{du}\log Det\, O_u = Tr\,\frac{dO_u}{du}\,O_u^{\;1}$$

Hereby we suppose u to range over operators in the same homotopy class. Furthermore we assume the trace-class condition

$$\left| Tr\,\frac{dO_u}{du}\,O_u^{\;1} \right| < \infty$$

Forman's theorem

(Invent. Math. 88, 447-493,1987.)

If A and B are elliptic boundary conditions on M with linear complement CA and CB

$$\frac{d}{du}\log\frac{Det\,O_{u,A}}{Det\,O_{u,B}}=\frac{d}{du}\log det\,\Phi_{O_u}$$

$$\Phi_{O_u}: CB \rightarrow CA \qquad\qquad \Phi_{O_u}=T_{CA}\,P_{CB}$$

P is the Poisson map of O whilst v is vector field on ∂M and transverse to it at each point

$$[T(f)](x):=\left(f(x),\frac{\partial}{\partial v}f(x),\ldots,\frac{\partial^{n\;\;1}}{\partial v^{n\;\;1}}f(x)\right)$$

Second order differential operators

Quadratic fluctuations in configuration space are governed by the Sturm-Liouville operator

$$D^2_{q_{cl}} L = O := \frac{d}{dt}\left(L_{\dot{q}\,\dot{q}} \frac{d}{dt} + L_{\dot{q}\,q} \right) + L_{q\,\dot{q}} \frac{d}{dt} + L_{q\,q}$$

The Poisson map $O\,P = 0, [T_{CB}(P)] = h_{CB}$ can be lifted to TM

$$\begin{bmatrix} \frac{d}{dt} & 1 \\ L_{q\,q} \quad \left(\frac{d}{dt} L_{\dot{q}\,q} \right) & \frac{d}{dt} L_{\dot{q}\,\dot{q}} \quad L_{\dot{q}\,q} + L_{q\,\dot{q}} \end{bmatrix} T(P) = 0$$

Bott pair description of boundary conditions

In d-dimensions, the Bott pair associated to some elliptic boundary conditions A is specified by two 2 d x 2 d matrices (A_0, A_1) with rank of $[A_0, A_1]$ equal 2 d

$$f \in A \quad \Leftrightarrow \quad \begin{bmatrix} [T f(t_0)] \\ [T f(t_1)] \end{bmatrix} = \begin{bmatrix} A_0 x \\ A_1 x \end{bmatrix}$$

The boundary conditions are self-adjoint if

$$A_0^\dagger J A_0 = A_1^\dagger J A_1$$

for J is the symplectic pseudo-metric.

Linear complement to the boundary conditions

$$A \sim (A_0, A_1)$$ Boundary conditions A

$$CA \sim (CA_0, CA_1)$$ Linear complement of A

$$(CA_0 A_0 + CA_1 A_1)x = 0$$

Examples:

$$Cauchy \sim (0, 1)$$

$$Periodic \sim (1, 1)$$

$$Dirichlet \sim \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right)$$

Forman's theorem for (second order) differential operators

$$\frac{Det\, O_{u_0,A}}{Det\, O_{u_0,B}} \frac{Det\, O_{u_1,B}}{Det\, O_{u_1,A}} = det \frac{[T_{CA}(P_{CB})_{u_0}]}{[T_{CA}(P_{CB})_{u_0}]}$$

$$[T_A(P_B)] := A_0[T(P_B)](t_0) + A_1[T(P_B)](t_1)$$

For Sturm-Liouville operators

$$Det\, O_{u_0,Cauchy} = Det\, O_{u_1,Cauchy}$$

Equivalence between T*M and TM descriptions

Fluctuations around the stationary phase approximations in phase and configuration space generate the same functional

$$x = R\tilde{x} \qquad R = \begin{bmatrix} 1 & 0 \\ L_{\dot{q}q} & L_{\dot{q}\dot{q}} \end{bmatrix} \qquad \begin{aligned} x &= (\delta q, \delta p) \\ \tilde{x} &= (\delta q, \delta \dot{q}) \end{aligned}$$

$$\begin{aligned} &det\left[R^{1}\left(CA_0 + CA_1 F_{u_1}(t_1, t_0)\right)R\right] \\ &= det\left[CA_0 + CA_1 F_{u_1}(t_1, t_0)\right] \end{aligned}$$

Examples of fluctuation determinant

Let F the fundamental solution of the linear dynamics
in phase space

$$\frac{Det\, O_{u_1, A}}{Det\, O_{u_0, A}} = det\, \frac{CA_0 + CA_1 F_{u_1}(t_1, t_0)}{CA_0 + CA_1 F_{u_0}(t_1, t_0)}$$

$$(CA_0, CA_1) \sim \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right)$$

Dirichlet b.c

1, 1right

$(CA_0, CA_1) \sim ¿$

Periodic b.c.

Determinant homotopy class
and
Morse index

If $D^2_{q_{cl}} L_B$ with b.c. B is self adjoint the spectrum is real. If the classical kinetic energy is strictly positive definite the spectrum is bounded from below.

$$sign\, Det\left(D^2_{q_{cl}} L_B\right) = e^{i\pi\mu_B}$$

$\mu_B =$ Number of negative eigenvalues

$\equiv$ Morse index for the b.c. B

Equal fluctuation determinants in T*M and TM

$$\mu_B = \eta_B$$

Intersection forms
and
Morse index

Duistermaat (Adv. in Math. 21, 173-195, 1976) : Morse index from intersections of Lagrangian planes.

Salamon & Zehnder (Com. in Pure and Appl. Math. XLV, 173-195, (1992) : (infinite dimensional) Morse theory for periodic solutions of Hamiltonian systems (Conley and Zehnder index).

Robbin & Salamon (Bull. LMS 27, 1995): spectral-flow of self-adjoint operators

Y. Long (Adv. in Math. 154, 76-131, 2000) : explicit iteration formulae for the Conley and Zehnder index in terms of the stability blocks of the linear flow.

Faddeev-Popov method

In the presence of zero modes of $D^2_{q_{cl}} L_B$ the stationary phase approximation can be applied only to fluctuations orthogonal to the nullspace. The constraint is imposed by writing

$$1 = \int \prod_\alpha d\tau_\alpha \delta\left(F_\alpha\right) \left| det \frac{\partial F_\beta}{\partial \tau_\gamma} \right|$$

$\tau_\alpha =$ Moduli of the invariant (sub)- group G

$$F_\alpha = \langle q \quad q_{cl,[\tau]}, \frac{\partial q_{cl,[\tau]}}{\partial \tau_\alpha} \rangle$$

Orthogonality condition

$$Tr\,\hat{K}(T) \approx e^{\frac{iS_{cl}}{\hbar}} \int \mu(dG) \int d[c] |det\, Z| \prod_n \delta(c_n) e^{\frac{i\delta^2 S}{\hbar}}$$

Coleman's regularisation method

The Morse index of $D^2_{q_{cl}} L_B$ is invariant under a sign definite small perturbation of the potential under which zero eigenvalues acquire a positive values $\lambda_n(\epsilon)$. If B stands for periodic b.c.

$$\lim_{\epsilon\downarrow 0}\left|\frac{Det\, D^2_{q_{cl}} L_{per.,\epsilon}}{det\, Z^2 \prod_{n\in ker} \lambda_n(\epsilon)}\right| = \left|det\, V\, det_{\perp}(1 \quad M)\right|$$

det V depends on the symmetry group and on the the parabolic (longitudinal) block of the monodromy matrix M. If the energy is the only conserved quantity

$$|det\, V| = \left|\frac{dT}{dE}\right|$$

Conclusions

- Forman's theorem offers a general dimensional reduction method to compute functional determinants of elliptic, not necessarily trace class operators.
- The computation of functional determinants through homotopy transformations naturally relates the Morse index (configuration space) to the Conley and Zehnder index (phase space) of periodic classical extremal.